Study of Topological Surfaces

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Abstract

This research project involves studying different topological surfaces, in particular, continuous, invertible functions from a surface to itself. Dealing with such functions on different topological surfaces requires developing ways of showing that two functions from a surface to itself are isotopic. The project focuses on a specific class of functions on surfaces, namely, the composition of twists along closed curves, and looks for explicit ways to explore new relations among such twists and to further simplify previously proven relations. Using labeled diagrams along with specified twists and closed curves, the braid relation, Lantern relation, and part of the Chain relation for a genus 2 surface have been proven. The search is now focused on finding twelve curves in a torus with nine boundary components such that the composition of twists along \(C_1, C_2, \ldots, C_{12}\) equals one right twist along each boundary of the nine-punctured torus. In order to save time and effort, a way of detecting the existence of boundary-interior relations has been studied.

The format of the paper goes as follows:

The paper starts with some definitions and terminology that clarify the notation to be used. Methodology is then explained. Following that is a method to determine when do boundary interior relations exist. Then, the Braid relation, the Lantern relation, and a part of the Chain relation for the genus 2 surface are proved. After that, there is a list of different configurations of curves on the nine-punctured torus that examines the possibility of existence of boundary interior relations. Finally, a brief reference about the future work on the project is mentioned.

Definitions and Terminology

Starting with terminology, topological surfaces are denoted by \(\Sigma\), curves are denoted by capital, small letters, or numbers \(A, B, C, \ldots, a, b, c, \ldots\), or \(1, 2, 3, \ldots\) and twists along curves are denoted by \(\tau\) or \(\delta\). So, if \(b\) is a curve then \(\tau_b\) is a twist along \(b\). Sometimes, a composition of twists along a certain curve is referred to by \(x\). All curves that are studied on different surfaces are always on the surface. That is, curves do not leave the surface. Moreover, going with convention, right twists are always applied to curves under study.

*A right twist, by definition, is a twist that results in a right turn along the curve under study.*

![Diagram of a right twist](image)
Definition: A function \( f: \Sigma \rightarrow \Sigma \) is said to be a homeomorphism if:

1. \( f \) is continuous.
2. \( f \) is invertible.
3. \( f^{-1} \) is continuous.

\( \text{Map}^+ (\Sigma) = \{ f: \Sigma \rightarrow \Sigma : \text{f is a homeomorphism and f is orientation preserving} \} \)

\( f_0 \) is said to be isotopic to \( f_1 \), denoted by \( f_0 \sim f_1 \) if there exists a continuous family of functions \( \{ f_t : t \in [0,1] \} \) in \( \text{Map}^+ (\Sigma) \) connecting \( f_0 \) to \( f_1 \).

**Theorem:** For a disk \( D \), all curves in \( \text{Map}^+ (\Sigma) \) are isotopic to the identity.

**Methodology**

To show that some \( f \) in \( \text{Map}^+ (\Sigma) \) is isotopic to the identity, we do the following:

1. Pick a collection \( C \) of arcs and circles in the surface \( \Sigma \) that cut \( \Sigma \) into a disk.
2. Draw \( f(C) \).
3. Show that \( f(C) \) can be continuously deformed back to \( C \).
   
   If so, then \( f \) is isotopic to the identity.

**When Can Boundary-Interior relations exist?**

Let \( \Sigma \) be any surface with curves \( C_1, C_2, \ldots, C_k \) that cut \( \Sigma \) up into \( n \) disjoint pieces \( P_n \), \( n = 1, 2, \ldots, n \) each of which has at least one boundary component from \( \Sigma \). Let the different pieces correspond to vertices on a separate graph \( G \), and let different curves correspond to edges on \( G \). Denote by \( g_n \), \( n = 1, 2, \ldots, n \) the genus of each piece, and by \( m \) the number of boundary components of each \( P_n \) minus the number of curves touching \( P_n \). Each vertex of \( G \) is labeled with a pair \((g, m)\). Next, note a matrix \( A \) of the graph \( G \) such that

\[
A_{ij} = \begin{cases} 
  m_i \text{ if } i = j \\
  \# \text{ of edges from vertex } i \text{ to vertex } j \text{ if } i \neq j 
\end{cases}
\]

Now, diagonalize the matrix by using elementary row/column operations. After each row operation, the same corresponding column operation must be performed. After diagonalizing \( A \), denote by \( b^+ \) be the number of positive entries on the diagonal.

If \( b^+ > 1 \) and for at least one of the vertices, \( m > 2g-2 \), then there does not exist a boundary-interior relation \( \delta = w \) where \( w \) is a word containing the twists \( \tau(C_1), \tau(C_2), \ldots, \tau(C_k) \). Otherwise, there might exist a boundary-interior relation.

An alternative way to determine \( b^+ \) is counting the number of positive eigenvalues for the matrix \( A \).
The Braid Relation

The Braid relation is: \( \tau_b \circ \tau_a \circ \tau_b = \tau_a \circ \tau_b \circ \tau_a \)

Let \( a \) and \( b \) be curves on surface \( \Sigma \) shown below.
Curves \( C \) and \( D \) cut surface \( \Sigma \) into a disk. To prove the Braid relation, we show

\[
\tau_b \circ \tau_a \circ \tau_b (C) = \tau_a \circ \tau_b \circ \tau_a (C) \tag{1}
\]
\[
\tau_b \circ \tau_a \circ \tau_b (D) = \tau_a \circ \tau_b \circ \tau_a (D) \tag{2}
\]

where \( \tau_a (C) \) and \( \tau_a (D) \) are right twists of curves \( C \) and \( D \) respectively along \( a \). Similarly, \( \tau_b (C) \) and \( \tau_b (D) \) are right twists of curves \( C \) and \( D \) respectively along \( b \).

Proof:
The Lantern Relation

The Lantern Relation is: $\tau_1 \circ \tau_2 \circ \tau_3 = \delta_1 \circ \delta_2 \circ \delta_3 \circ \delta_4$

Let $\Sigma$ be a disk with three holes in it. Curves A, B, and C, shown below, cut $\Sigma$ into a disk. $\tau_1$, $\tau_2$, $\tau_3$, $\delta_1$, $\delta_2$, $\delta_3$, and $\delta_4$ are right twists applied to curves A, B, and C. To prove the Lantern Relation, we must show

$\tau_1 \circ \tau_2 \circ \tau_3 (A) = \delta_1 \circ \delta_2 \circ \delta_3 \circ \delta_4 (A)$ and,
$\tau_1 \circ \tau_2 \circ \tau_3 (B) = \delta_1 \circ \delta_2 \circ \delta_3 \circ \delta_4 (B)$ and,
$\tau_1 \circ \tau_2 \circ \tau_3 (C) = \delta_1 \circ \delta_2 \circ \delta_3 \circ \delta_4 (C)$

Proof:
THE LANTERN RELATION

Cut A, B, and C transform the figure into a disk.
$$\tau_1 \circ \tau_2 \circ \tau_3 (A) = \delta_1 \circ \delta_2 \circ \delta_3 \circ \delta_4 (A)$$
\[ s_4(B) = s_3 \circ s_4(B) = s_2 \circ s_3 \circ s_4(B) \]

\[ s_1 \circ s_2 \circ s_3 \circ s_4(B) \]

\[ s_1 \circ s_2 \circ s_3 \circ s_4(B) = s_1 \circ s_2 \circ s_3 \circ s_4(B) \]
Chain Relation for Genus 2 Surface

The Chain Relation for a Genus 2 surface is: 

\[(τ_1 \circ τ_2 \circ τ_3 \circ τ_4)^{10} = I\]

Let \(Σ\) be a genus 2 surface. Curves A, B, and C, and D, shown below, cut \(Σ\) into a disk. 

\(τ_1, τ_2, τ_3,\) and \(τ_4\) are right twists applied to curves A, B, and C. To prove the Chain Relation for a genus 2 surface, we must show

\[(τ_1 \circ τ_2 \circ τ_3 \circ τ_4)^{10}(A) = A\] and,

\[(τ_1 \circ τ_2 \circ τ_3 \circ τ_4)^{10}(B) = B\] and,

\[(τ_1 \circ τ_2 \circ τ_3 \circ τ_4)^{10}(C) = C\] and,

\[(τ_1 \circ τ_2 \circ τ_3 \circ τ_4)^{10}(D) = D\]

Proof:

For the purpose of simplicity, let \(x = τ_1 \circ τ_2 \circ τ_3 \circ τ_4\)
\[(\tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_4)^{10} (B) = B \], \[(\tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_4)^{10} (C) = C \], and \[(\tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_4)^{10} (D) = D \] are not yet proven. They will be proved as part of the future work on the project.

**Different Configurations of Curves on a Nine-Punctured Torus**

In personal communication, Dr. David Gay explained that there should exist twelve curves \(C_1, C_2, \ldots, C_{12}\) in a torus with nine boundary components such that the composition of twists along \(C_1, C_2, \ldots, C_{12}\) equals one right twist along each boundary of the nine punctured torus. Since we are looking for boundary interior relations, we have tried to avoid the use of “hopeless” curves. Hence, the following list has been created in order to determine which configuration of curves on the nine-punctured torus might work.
\[
A := \begin{bmatrix}
-1 & 2 \\
2 & 6 \\
\end{bmatrix}
\quad
2R_1 + R_2 := \begin{bmatrix}
-1 & 2 \\
0 & 10 \\
\end{bmatrix}
\quad
2C_1 + C_2 := \begin{bmatrix}
-1 & 0 \\
0 & 10 \\
\end{bmatrix}
\]

\( b^+ = 1 \), so there might exist a boundary-interior relation \( \delta = w \) where \( w \) is a word containing the twists \( \tau(C_1) \) and \( \tau(C_2) \)
\[ A := \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \quad -R_1 + R_2 \quad \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \quad 2C_1 + C_2 \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \]

\( b^+ = 2 > 1 \) and vertex A has \( m = 2 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.

\( b^+ = 1 \), so there might exist a boundary-interior relation.

\( b^+ = 2 > 1 \) and the right vertex has \( m = 1 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.
$b^+ = 2 > 1$ and the upper vertex has $m = 1 > 2g - 2 = -2$, so there does not exist a boundary-interior relation.

$b^+ = 2 > 1$ and the upper vertex has $m = 2 > 2g - 2 = -2$, so there does not exist a boundary-interior relation.

$b^+ = 2 > 1$ and the right vertex has $m = 2 > 2g - 2 = -2$, so there does not exist a boundary-interior relation.
\( b^+ = 2 > 1 \) and the upper vertex has \( m = 3 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.

\( b^+ = 0 \), so there might exist a boundary interior relation.

\( b^+ = 2 > 1 \) and the upper left vertex has \( m = 1 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.
For matrices with dimensions higher or equal to 5x5, \( b^+ \) is determined by the number of positive eigenvalues on the diagonal of a matrix. This can be easily done using the Maple Software. It is rather easier to find \( b^+ \) in this manner than diagonalizing the matrix via row/column operations.

\[ b^+ = 3 > 1 \] and the upper left vertex has \( m = 0 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.

\[ b^+ = 3 > 1 \] and the lower left vertex has \( m = 0 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.

\[ b^+ = 3 > 1 \] and the lower left vertex has \( m = 0 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.
\( b^+ = 3 > 1 \) and the lowest vertex has \( m = 0 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.

\( b^+ = 3 > 1 \) and the lower left vertex has \( m = 0 > 2g - 2 = -2 \), so there does not exist a boundary-interior relation.
\[
A := \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]

Eigenvalues are:

\([-2.927470183, -2.414213564, -2.203312892, -1.000000001, -.7673388740, .4142135623, .6448503424, 1.253271606]\)

\(b^+ = 3 > 1\) and all the vertices have \(m = -1 > 2g - 2 = -2\), so there does not exist a boundary-interior relation.

12 Curves on a Nine-Punctured Torus

The following twelve curves \(C_1, C_2, \ldots, C_{12}\) have been tried on a torus with nine boundary components hoping that the composition of twists along \(C_1, C_2, \ldots, C_{12}\) equals one right twist along each boundary of the nine punctured torus.

The presented configuration does not work because of a simple observation in Figure 2. The lower part of the vertical line in Figure 2 cannot be further simplified because \(\tau_i(C), i = 7, 8, 9, 10, 11, 12\) do not cross it. Consequently, the whole configuration does not eventually simplify to curve \(C\) after performing the twelve twists.

Because of the observation made, we could have stopped drawing at Figure 2. However, the figures following Figure 2 were just done for confirmation.
Future Work on the Project

The future work on the project will be primarily focused on finding boundary-interior relations for different surfaces as well as proving these relations by means of diagrams similar to the ones already used. The main interest will be finding twelve curves $C_1$, $C_2$, ...
..., $C_{12}$ in a torus with nine boundary components such that the composition of twists along $C_1$, $C_2$, ..., $C_{12}$ equals one right twist along each boundary of the nine punctured torus.