Theoretical Proof of Lehmer's Conjecture for Polynomials of Small Degree

Undergraduate Research Project
Professor: Daniel Madden
Undergraduate Researcher: Jad C Halimeh

Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

be a polynomial with integer coefficients. Another way to write this polynomial is:

$$f(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

Then the Mahler's measure of f is simply the product of the absolute value of a_n and the absolute values of all the roots of $f(x)$ that are outside the unit circle. In formula, this is expressed as:

$$M(f(x)) = |a_n| \prod_{i=1}^{n} max\{1, |r_i|\}$$

The Mahler's measure satisfies the following properties:
1. $M(f(x)g(x)) = M(f(x))M(g(x))$
2. $M(f(-x)) = M(f(x))$
3. $M(x^n f(\frac{1}{x})) = M(f(x))$

Now for the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

we get a Mahler's measure $C = 1.1762808182599\ldots$.

Lehmer's conjecture states that:
Given $f(x)$ a polynomial with integer coefficients, then if $M(f(x)) > 1$, this implies that $M(f(x)) \geq C$

or equivalently,

Given $f(x)$ a polynomial with integer coefficients, then if $M(f(x)) < C$, this implies that $M(f(x)) = 1$.

Two more properties that might help us in proving the conjecture are:
4. If $M(f(x)) = 1$, then $f(x)$ is a factor of $x^k - 1$ for some integer k
5. If $f(x) \neq x$ is an irreducible polynomial and not a factor of $x^k - 1$ for any integer k, then $M(f(x)) < 1.4$ implies that the coefficients of $f(x)$ are symmetric, the same forward or backward.

There have been several numerical searches for polynomials with small Mahler's measure, and for small degree the searches are exhaustive. Thus, the smallest Mahler's measure for each degree up to 24 is known.

The research project proposed here is to find a new *theoretical* proof that Lehmer's conjecture holds for polynomials of degree $n \leq 10$.