The Propagation of Waves through a Cracking Whip

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Abstract

The properties involved in a cracking whip are explored. The goal is to provide a starting point for the exploration of this phenomenon. Eventually, scientists would like to produce an equation which describes the shape of the whip with respect to space and time. Some preliminary results are discussed in this paper.

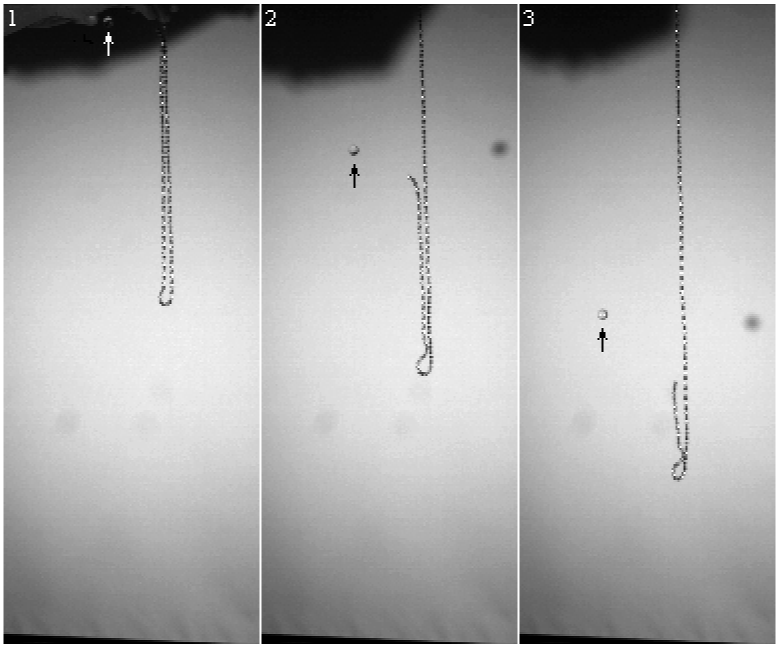
Introduction

When a whip is cracked, the user moves his or her arm back and forth, which imparts energy on one end of the whip. This back and forth motion of the experimenter's arm creates a kinetic wave in the material. This wave propagates through the whip and by the time it reaches the tip of the whip, that tip is moving at supersonic speeds. This may be curious to some since there is no way the user can move his or her arm at the speed of sound. It may be known why the velocity of the tip is so much higher than the original velocity, but it is not quite known how the shape of the whip changes over time, and how this change is related to the initial velocity. It is hoped that by studying related phenomena, some insight will be gained on how this occurs.

Theory

One related phenomenon that is helpful for understanding why the whip cracks is demonstrated by a falling chain. If one end of a chain is fixed at a certain height, and the other end is held at that same height, a U-shaped curve is formed(fig. 1.1). One might expect that when the free end is dropped it will accelerate normally due to gravity. According to Newton, all free falling objects accelerate at the same rate, which is known to be 9.8 m/s^2. However, when this experiment is performed the free end reaches a point where it is actually accelerating faster than this (fig. 1.2 and 1.3). This is because when an object is suspended it has initial potential energy mgh. When it is dropped and reaches the bottom of its fall, it has zero potential energy and kinetic energy $\frac{1}{2}mv^2$. This energy must be conserved, however, and in the case of the chain, the mass of the moving section is constantly decreasing. To compensate and maintain the same energy, the velocity is higher than if the mass were constant, as in a free falling object.

This is also why a whip cracks. When the user initially lashes it, the whip has some curvature due to the motion of the arm (fig. 2 and 3.1). The whip can be thought of as two parts, separated by a sharp curvature (fig. 2, fig. 3.1, and fig. 4). The side toward the tip (fig. 2.2) has an initial energy, because of the upward motion placed on it by the user. However, as the curve moves further toward the tip (fig 3.2, 3.3 and 3.4), that side has an decreasing mass, and compensates by an increase in velocity. This is how it can reach supersonic speeds.



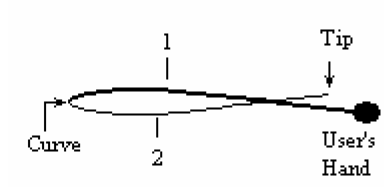


Fig. 2. Diagram showing how the components of the whip are broken up. **1.** The portion of the whip manipulated by the user. **2.** The portion of the whip toward the tip. As 1 becomes smaller, and 2 becomes larger, 1 must increase its velocity to conserve energy. The whip "cracks" just before 1 reaches zero..

Fig. 1. Images from the chain-drop experiment. **1:** Initial position of the chain. **2:** 159 ms after release of the chain. **3:** 214 ms after release of the chain. An accompanying ball and the tip of the chain were released at same time. It can clearly be seen that the tip of the chain falls faster than the free falling ball.

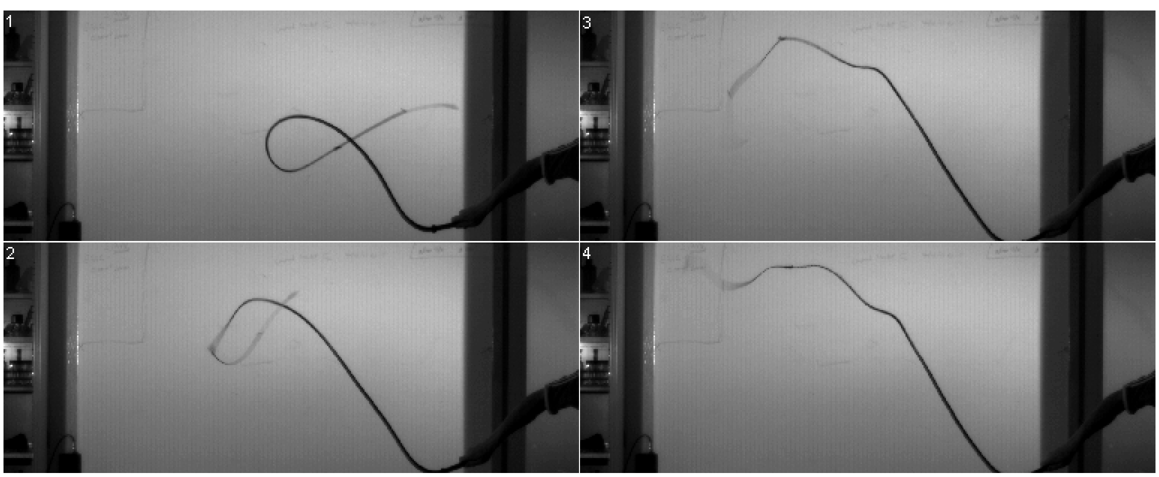


Fig. 3. Images of a whip being cracked. **1:** The initial curve given by the user's rapid back and forth motion can be seen at 44.5 ms. **2:** The curve has moved out toward the end of the whip, and the portion of the whip that is on the outside of this curve has decreased at 56.5 ms. **3**: This is just before the curve reaches the tip of the whip at 62.5 ms. **4:** When the curve does reach the tip, supersonic velocities occur and the whip cracks 65.5 ms after the initial motion by the user.

The following model of a whip comes from Szabo I (1972) Hohere Technische Mechanik. Springer, Berlin, Heidelberg, and New York, 5th ed., pp 131-132. It models the whip in one dimension (fig. 4). Assume an arbitrary length x, which represents any

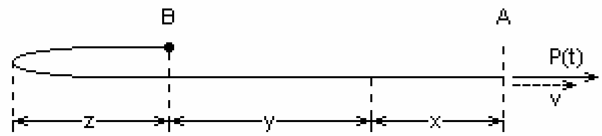


Fig. 4. A diagram of the simplified model of a whip. Point **A** represents the end of the whip that is pulled by the user with force P(t) at velocity v. Point **B** represents the tip of the whip. **x** represents a fixed point on the whip toward the handle. **y** represents an expanding section of the whip from x to a point that is at the same position as the tip. **z** represents a shrinking section of the whip that is the size of the arc from the tip to the curve. The total size of the whip is x + y + 2z.

fixed portion on the whip, close to the handle. There is also a changing distance y, which becomes longer as z becomes smaller. In other words, as the curve moves further toward the tip, z becomes smaller and y becomes larger. The total length of the whip, is represented by $l = x + y + 2z$. Kinetic energy is then calculated by $E = \frac{\mu}{2}(x + y + z)\dot{x}^2 + \left(\frac{\mu}{2}z + \frac{m}{2}\right)\dot{y}^2$, or its more useful form: $E = \frac{\mu}{2}(l - z)\dot{x}^2 + \frac{1}{2}(\mu z + m)(\dot{x} + 2\dot{z})^2$ where μ is the linear density of the material and m is a small mass placed at point B. The second is more useful, because x and z are the coordinates of interest. This is simply $\frac{1}{2}mv^2$, where mass is μ times the length. The first half of the equation is the energy of the whip on the side of the curve closer to the handle; the second part is the energy on the side closer to the tip. They must be handled differently since they are moving at different velocities. Note that if m is zero, a singularity arises later in the equation. Potential energy is given by $U = -P(t)x$, where P(t) is the force pulling on the whip as a function of time. The Lagrangian of this system is E – U, so using x and z as coordinates, the Lagrangian equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$ becomes $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}}\right) = \frac{\partial E}{\partial x} - \frac{\partial U}{\partial x}$ since U has no $\dot{x}$ dependence. Similarly, $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{z}}\right) = \frac{\partial E}{\partial z} - \frac{\partial U}{\partial z}$. Plugging in the equations for kinetic and potential energy gives $\ddot{x}(\mu l + m) + 2\left[\mu\dot{z}^2 + \ddot{z}(\mu z + m)\right] = P(t)$, and $\mu\dot{z}^2 + (\mu z + m)(\ddot{x} + 2\ddot{z}) = 0$. For simplicity, it can be assumed that the velocity v at which the whip is pulled is constant, this makes the equations $2\mu\dot{z}^2 + 2\ddot{z}(\mu z + m) = P(t)$, and $\mu\dot{z}^2 + 2\ddot{z}(\mu z + m) = 0$, because the acceleration of x is zero. By rearranging the second equation into $\frac{\mu\dot{z}}{\mu z + m} + 2\frac{\ddot{z}}{\dot{z}} = 0$, it can be seen as the time derivative of $\ln(\mu z + m) + 2\ln\dot{z}$, which is equal to zero. This differential equation, combined with the initial conditions $x(0) = 0$, $y(0) = y_0$, $\dot{y}(0) = 0$, $z(0) = \frac{1}{2}[l - y(0) - x(0)] = \frac{1}{2}(l - y_0) = z_0$, and $\dot{z}(0) = -\frac{1}{2}[\dot{y}(0) + \dot{x}(0)] = -\frac{v}{2}$ give $\dot{z} = -\frac{v}{2}\sqrt{\frac{\mu z_0 + m}{\mu z + m}}$, which yields the equation for z as a function of time

$z(t) = \frac{1}{\mu}\left[\sqrt[3]{(\mu z_0 + m)\left(\mu z_0 + m - \frac{3\mu}{v}vt\right)^2} - m\right]$. It is this function that will be examined in future research to determine how well the photographs match this function. This function shows how fast the wave will travel through the whip, not how fast the tip moves. In future work, a function for $\dot{y}$ will be investigated to determine the speed of the whip. Also, the equations will be solved for P(t) to determine the force needed to maintain a constant velocity, or how things will turn out if velocity is not constant, but P(t) is given.

The subject of this research is to study the shape and propagation of that curve. To do this, however, the initial velocity must be known. For later research, it will be helpful to know what the initial curvature is. To do this, a simulation is required. A machine was designed to give reproducible results. Thus, an initial curvature may be obtained, and the initial velocity known. The machine that was designed to pull the whip used a rat trap. As a result, a large part of the current research went in to studying the physics of a rat trap.

Physics of the Rat Trap:

Fig. 5. The rat trap. **A.** The U shaped bar. **B.** the two springs that power the rat trap. **C.** The 'holding bar'. The U shaped bar is pulled back 180 degrees, then the holding bar is attached to the small rectangular fixture. When the fixture is disturbed, the holding bar is freed, and the trap is sprung.

The rat trap (fig.5) is comprised of two springs (B) that move a 'U shaped' bar (A) through a 180º rotation. The initial goal was to find out what the velocity of the bar would be at any given point. The conservation of rotational energy is what motivates the following discussion. Rotational kinetic energy is the sum of all torques, so if torque were known as a function of angle, it would be a simple matter of integrating that function to get rotational energy, $\frac{1}{2}I\omega^2$. In theory, torque should be a linear function of angle, however in reality the springs buckle slightly, giving what would appear to be a curved function. Instead, a chart can be set up, approximating the angular acceleration at each interval where torque is known to give a left or right sum.

The way torque was calculated at each point, using the equation $\tau = rF$. Then the angular acceleration at each point was calculated using $\alpha = \frac{\tau}{I}$, where I is the moment of inertia. Once an angular acceleration was calculated for each point, the angular velocity was calculated using the kinematic equation $\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$. In this way, the angular acceleration for each interval can be approximated by the angular acceleration at the beginning of the interval. This gives a left sum approximation for the system at each point (see table 1.).

Once the kinematic equation is solved for angular velocity, it is a simple matter of multiplying by the length of the arm to get linear velocity at the tip.

Moment of inertia is calculated by assuming that the cross bar is a point mass, and the other two bars are rods rotating about an axis at one end. This gives a formula for moment of inertia $I = \frac{2}{3} M_{Leg} L^2 + M_{Cross} L^2$.

Methods

First, images were shot of a chain being dropped as described in the theory section. One end of the chain was tied to the ceiling, and the other suspended next to it. There was a small free falling ball one or two centimeters in diameter dropped from the same level as the end of the chain. They were both dropped at the same instant, so the difference in acceleration could be observed.

Fig. 6. Initial method for performing the string whipping experiment. The user's hand can be seen holding a small handle, which was used to pull the string downward. The other end of the string moves upward rapidly, creating the crack. The 3-centimeter bar was drawn in for visibility. The user imparts a downward velocity of about 10 to 12 meters per second.

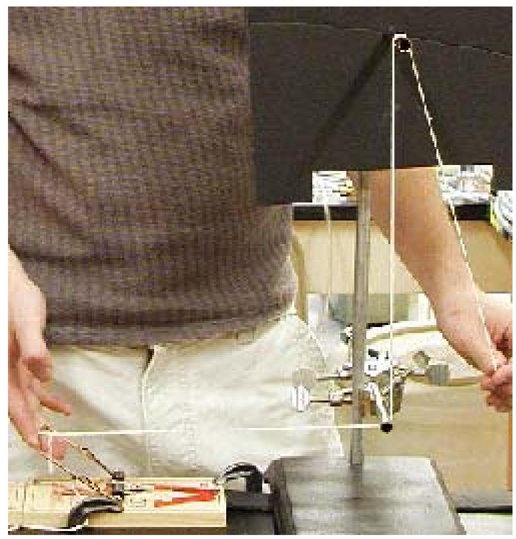


Fig. 7. Initial rat trap experiment. The string was tied to the bar of the Rat Trap, and was wrapped around two other bars. This created the cracking action

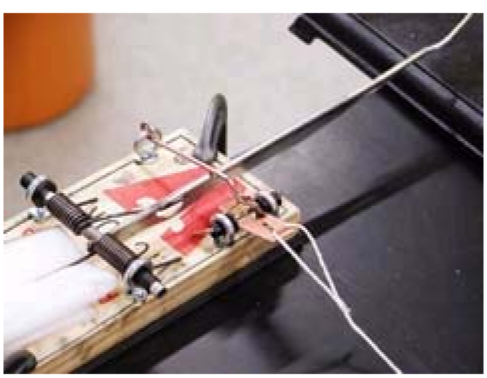


Fig. 8. A modified Rat Trap. The U shaped bar has been replaced with a single, 21 cm bar. A single spring was used.

To obtain images of the whipping action, a bar 3 centimeters in diameter was fixed at about 2 meters off the ground, and different lengths of kite string ranging from 50cm to 150 cm were used to simulate the whip. The string was tied to a short handle at one end, to be manipulated by the experimenter and the other end was draped over the bar (fig. 6). The user rapidly jerked the handle downward, and the upward motion of the free end created the crack. In fact, the crack could even be heard from this string. This data was captured in multiple high speed video files. There were also several high speed videos taken with the rat trap propelling the string. High speed videos of the actual whip being cracked were taken by illuminating the marker board with high powered lights, and having the human stand in front of the marker board and cracking the whip. The camera captures the silhouette of the whip. These videos were all taken at rates of 1000 or 2000 fps, with shutter speeds ranging from 1/2000 to 1/8000.

At first, the rat trap was clamped face up on the table (fig. 7). Two small horizontal bars about 1 centimeter in diameter were fixed near the trap. The string was tied to the end of the rat trap, fed under the lower bar, and over the higher bar. When the trap was triggered, it pulled the string but because the rat trap was perpendicular to the bars, the resulting curve contained an artifact. For this reason, the design was adjusted such that the trap was placed vertically and its bar moved toward the floor. The trap was clamped to a solid metal plate, which was clamped to the leg of the table. One bar was eliminated from the design (fig. 9), and the shape of the curve was much better. The design of the trap itself was also altered in an attempt to raise the velocity of the pull. The U shaped bar, was replaced with a single rod approximately 21 centimeters in length, but only one spring was used to propel it (fig. 8).

Fig. 9. Alterations made to the mousetrap experiment. The modified rat trap has been repositioned so that it now pulls parallel to the bar suspending the string. This creates a whipping action with a better shape.

Methods for doing Rat Trap Measurements:

The force at each point along the path traveled by the bar was obtained by clamping the device to a table, and using a scale to measure how much force was required to maintain the arm at each angle. A protractor was used to measure the angle, and all measurements were taken perpendicular to the bar, so a cross product was not needed.

The length of the bar was measured by placing a kite string against the bar, and measuring the length of the string. The bar was weighed on a normal balance scale. To obtain the mass of any section of the bar, the mass per unit length was known and could be multiplied by the length of that section. The length of each unit was measured with a caliper.

The moment of inertia was calculated in this way. When the mass of a leg was needed it was calculated by multiplying the mass per unit length times the length of that leg. A similar method was used for the mass of the crossbar.

Results

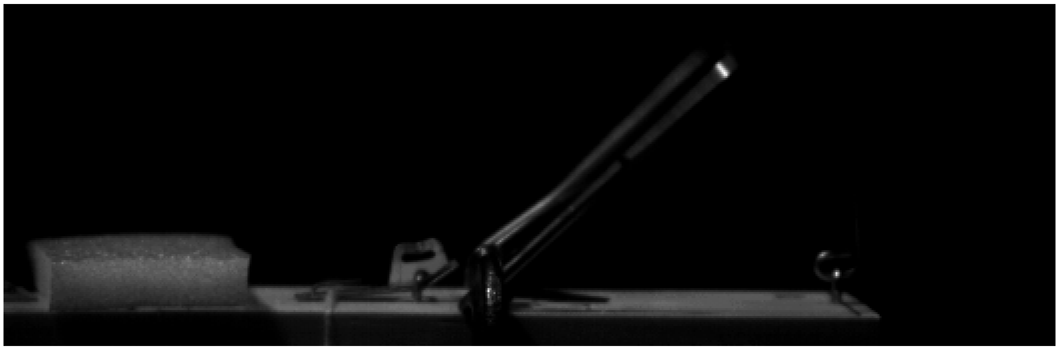


Fig. 10. This is the conventional rat trap in the process of being triggered. The tip of the bar is traveling at about 15 meters per second, about 25 milliseconds after it was triggered. By the time it reaches the shock absorber (left) at the end the tip will be moving about 30 meters per second.

angle	kg stress	N stress	tque(rF)	alpha(t/I)	d(angle)	d(vel)	vel(ang)^2	vel(ang)	v(lin)
0	3	29.4	2.41962	38751.1	0	0	0	0	0
75	2.3	22.54	1.85504	29709.2	1.309	77778.5	77778.5	278.888	22.9553
90	2	19.6	1.61308	25834.1	0.2618	13526.7	91305.2	302.167	24.8714
105	1.6	15.68	1.29046	20667.3	0.2618	10821.4	102127	319.572	26.304
120	1.4	13.72	1.12916	18083.9	0.2618	9468.69	111595	334.059	27.4964
135	1.25	12.25	1.00818	16146.3	0.2618	8454.18	120049	346.481	28.5189
150	1.1	10.78	0.88719	14208.7	0.2618	7439.68	127489	357.056	29.3893
165	1	9.8	0.80654	12917	0.2618	6763.35	134252	366.405	30.1588

Table. 1. These are the calculated values for the velocity of the trap at different points throughout the rotation.

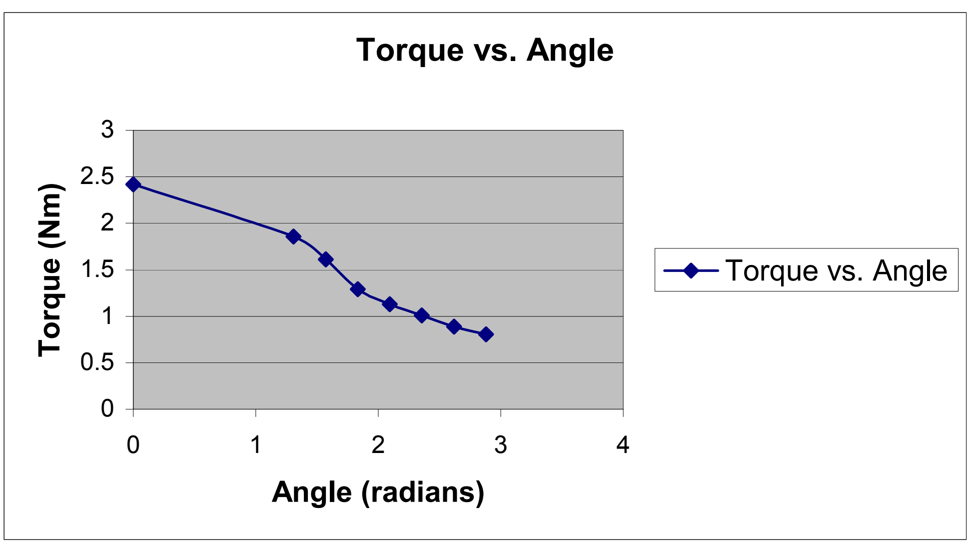


Fig. 11. A graph of the torque of the rat trap. The torque of the rat trap was measured every fifteen degrees through π radians of rotation.

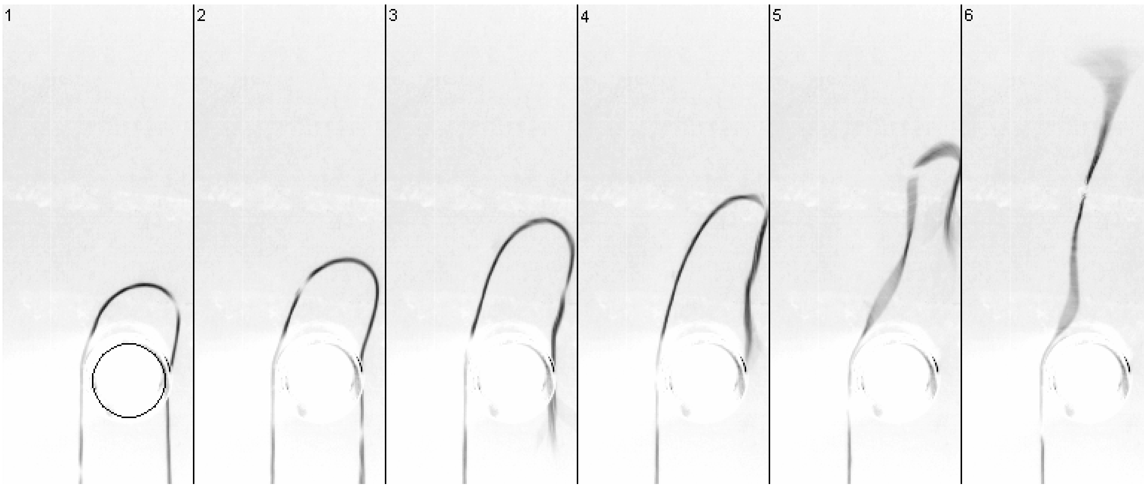


Fig. 12. These are six images of the string being whipped, which nicely show the shape of the whip. The colors have been inverted for visibility, and the bar has been drawn in frame 1. At this frame rate, any approximation for the velocity of the tip of this string would not be accurate, however in frame 6, the tip is probably moving at about the speed of sound. These are taken at 21.1, 21.2, 23.1, 23.6, 24.1, and 24.6 ms after the string began to move.

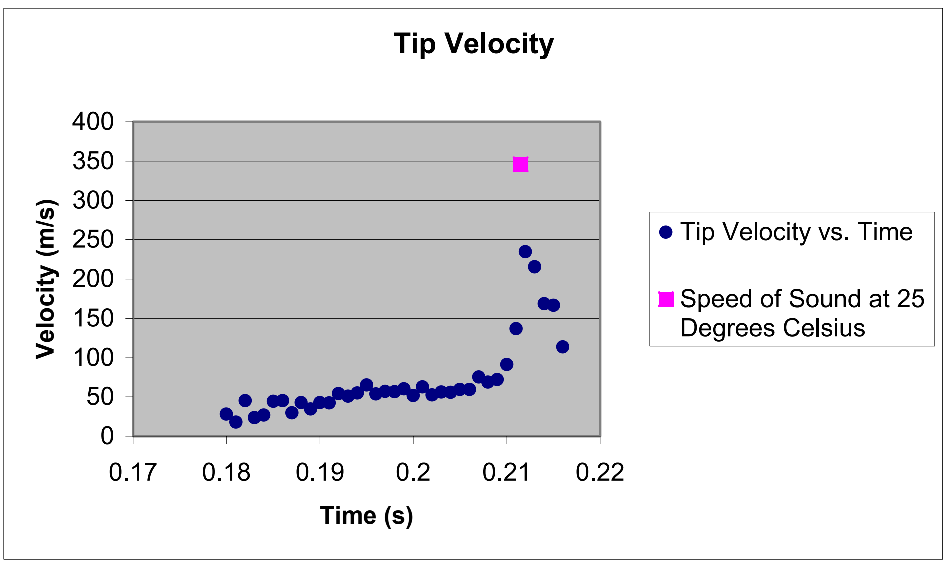


Fig. 13. A graph of the whip's velocity at the tip. The Square dot represents the speed of sound. While the data did not show it due to the low sampling rate, the whip reached the speed of sound at about 211 milliseconds after the hand began moving**.**

Discussion

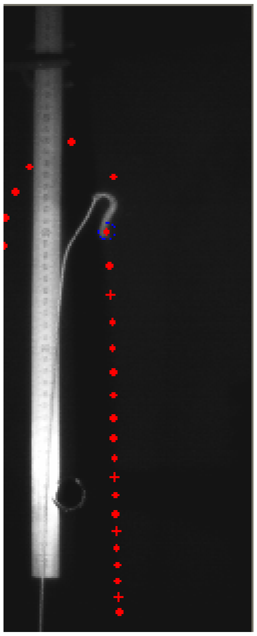


Fig. 14. An analysis of a video clip. This is one frame of a high speed video file of a string being cracked like a whip. The farther apart the dots are, the faster it is moving. This was taken at 2000 frames per second, which means each dot represents the string's movement in 1/2000 of a second

When the string was first cracked using a human hand, the shape of the curve was very good, however, there was too much variation from one trial to the next in the speed the string was jerked. In addition, it was difficult to image the appropriate section of the whip reproducibly. For this reason, the rat trap is needed for reproducibility and so the speed can be calculated. At first, the shape of the curl was very distorted, and the results were not very valuable, however, after making several adjustments to the design, including adding a single bar, which is longer, and changing the angle at which the trap pulls the string, a very good result was obtained.

Rat Trap:

The interesting thing about this rat trap was not how far the calculations were from the actual speed of the bar at the end, but rather how close they were. The reason that they should not be close is that when the final speed of the rat trap was estimated, the method used was to calculate the angular acceleration at different intervals and to assume that acceleration is constant through the interval. It should have estimated significantly low because the acceleration was not constant but increasing.

Future research:

Now that reasonable quality images have been made of the string whip, further analysis should determine how well they fit into the equations. This new work will take place beginning this summer will continue until completion.

The rat trap needs to be further analyzed to see if it more closely models the approximation if it is set off differently, namely without the 'holding bar'.

Velocity of the rat trap needs to be determined as a function of time, instead of a function of angle. This way when the video of the string is analyzed, it can be determined at each time interval what the velocity is.

Appendix A:

Using High Speed Photography in Scientific Research.

An important part of doing this experiment well was knowing how to use the camera, lights, and computer software to capture the best images possible. With such sophisticated equipment, learning to use it properly was no small task. To do this, several images were taken of other high speed phenomena that didn't relate to this particular experiment per se, but yielded very good, interesting results. Several images were taken of a small rubber popper. This popper is placed on a table, and when it inverts the force causes it to leap off the table. Also observed were fog rings produced by a small toy fog ring maker (fig. 15).

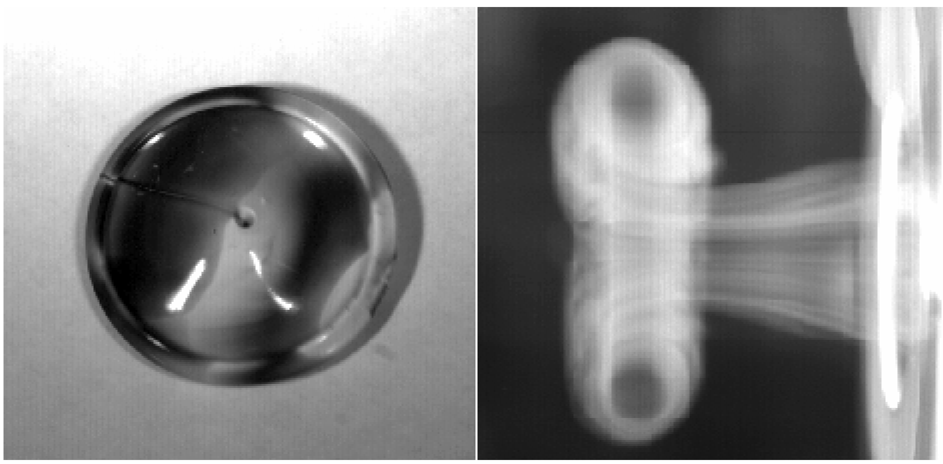


Fig. 15. High speed images taken during the course of this experiment to understand high speed.

It can be very difficult to capture an image just right. There are many factors that must be just right in order to capture good images. The lighting has to be at the right intensity and angle so that the image can be seen, but glare is not present. These lights are so powerful, that they can not be left on very long. They will begin to burn or melt an object on which they are shining. These bulbs also have a short life span, and must be preserved. Also involved are setting the focus and aperture. This must be done while the lights are on, so that the image can be seen, but as mentioned, the lights must be used extremely sparingly. The considerable amount of time spent during the course of this experiment simply learning the techniques involved proved to be an invaluable part of collecting data for the actual experiment.