Research Report: Mahler's Measure and Lehmer's Conjecture

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1 INTRODUCTION

Consider a polynomial $f(x)$ with integer coefficients:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

This can be rewritten as the following:

$$f(x) = a_n(x - \alpha_n)(x - \alpha_{n-1}) \cdots (x - \alpha_1),$$

where α_1, α_2, ..., α_n are the roots of $f(x)$.

Mahler's measure $M(f)$of $f(x)$ is the product of the absolute value of a_n with the magnitudes of those roots of $f(x)$ that are outside the unit circle:

$$M(f) = |a_n| \cdot \prod |\alpha_i| \text{ such that } |\alpha_i| > 1.$$

Lehmer's conjecture says that for any integer coefficient polynomial, its Mahler's measure is either 1, C, or greater than C, where C is Mahler's measure for the polynomial:

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 = 0.$$

2 Real, Distinct Cubic Roots

We begin approaching this problem by solving for the real, distinct roots of the cubic equation and this is what we will discuss here.

Consider a cubic of the form

$$x^3 + ax^2 + bx + c = 0,$$

and let y be such that

$$x = y + k.$$

Using Taylor's formula, we get:

$$f(y + k) = f(k) + f'(k)y + \tfrac{f''(k)}{2}y^2 + \tfrac{f'''(k)}{6}y^3 = 0,$$

and from the equation for $f(x)$, we know that:

$$f(k) = k^3 + ak^2 + bk + c,$$

$$f'(x) = 3x^2 + 2ax + b \Longrightarrow f'(k) = 3k^2 + 2ak + b,$$

$$f''(x) = 6x + 2a \Longrightarrow \frac{f''(k)}{2} = 3k + a, \text{ and}$$

$$f'''(x) = 6 \Longrightarrow \frac{f'''(k)}{6} = 1.$$

Substituting these values in the equation for $f(y+k)$, we get:

$$y^3 + (3k+a)y^2 + (3k^2 + 2ak + b)y + (k^3 + ak^2 + bk + c) = 0.$$

Now to make this equation look better and simpler to play with, let us set

$$k = -\frac{a}{3},$$

in which case our equation becomes:

$$y^3 + (\frac{a^2}{3} - \frac{2a^2}{3} + b)y + (-\frac{a^3}{27} + \frac{a^3}{9} - \frac{ba}{3} + c) = 0$$

$$y^3 + py + q = 0,$$

where $p = b - \frac{a^2}{3}$ and $q = c - \frac{ba}{3} + \frac{2a^3}{27}$. Now we can begin to analyze this equation, and find, based on the values of p and q, real, distinct roots for our cubic. Note that it is very easy to find the roots x_1, x_2, and x_3, once you find the roots y_1, y_2, and y_3 as $x_i = y_i - \frac{a}{3}$, $i = 1, 2, 3$.

Let $y = u + v$. This means that

$$(u+v)^3 + p(u+v) + q = 0$$

$$\Longleftrightarrow u^3 + v^3 + 3uv(u+v) + p(u+v) + q = 0$$

$$\Longleftrightarrow u^3 + v^3 + (3uv + p)(u+v) + q = 0.$$

Now, this equation is indeterminate (impossible to solve) without having another relation between u and v. Now as we have an infinite number of combinations of u and v such that $y = u + v$ for a given value of y, we choose that combination which in addition satisfies the relation:

$$uv = -\frac{p}{3},$$

in which case the cubic becomes:

$$u^3 + v^3 + q = 0$$

$$\Longleftrightarrow u^3 + v^3 = -q.$$

Moreover, $uv = -\frac{p}{3} \Longrightarrow u^3v^3 = -\frac{p^3}{27}$, so we get two equations with two unknowns, the latter being u^3 and v^3, as follows:

$$u^3v^3 = -\frac{p^3}{27},$$

$$u^3 + v^3 = -q.$$

Now, $u^3 + v^3 = -q \implies u^3 = -v^3 - q$, and this along with $u^3v^3 = -\frac{p^3}{27}$

$$\implies (-v^3 - q)(v^3) = -\frac{p^3}{27}$$

$$\iff (v^3)^2 + qv^3 - \frac{p^3}{27} = 0,$$

which is just a quadratic equation. Due to the symmetry of the system of two equations in u^3 and v^3, one of the two roots of this quadratic will be that of v^3, which we shall call B, while the other will be that of u^3, which we shall call A. The discriminant of this equation is $\delta = q^2 + \frac{4p^3}{27}$, giving the following:

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},$$

$$\text{and } B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Now, as $u^3 = A$, we get the following values for u:

$$u = \sqrt[3]{A},$$

$$u = \omega\sqrt[3]{A},$$

$$u = \omega^2\sqrt[3]{A},$$

where ω is the complex root of unit, $\omega = \frac{-1+i\sqrt{3}}{2} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})$. Similarly, we get the following values for v:

$$v = \sqrt[3]{B},$$

$$v = \omega\sqrt[3]{B},$$

$$v = \omega^2\sqrt[3]{B}.$$

Note here that every value of v does *not* correspond to all the values of u, as the relation $uv = -\frac{p}{3}$ must hold. Thus, remembering that $y = u + v$, we get the following y roots for our cubic:

$$y_1 = \sqrt[3]{A} + \sqrt[3]{B},$$

$$y_2 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B},$$

$$y_3 = \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}.$$

Now in our research we are interested only in the case when we get three *distinct, real* roots for our cubic. This is possible only when $\Delta < 0$, where:

$$\Delta = 4p^3 + 27q^2,$$

in which case A and B will be complex conjugates taking on the following forms:

$$A = -\tfrac{q}{2} + i\sqrt{-\tfrac{\Delta}{108}},$$

$$B = -\tfrac{q}{2} - i\sqrt{-\tfrac{\Delta}{108}}.$$

Now as Δ is strictly negative, $\sqrt[3]{A}$ and $\sqrt[3]{B}$ will surely be complex, and having the following form:

$$\sqrt[3]{A} = \alpha + \beta i,$$

$$\sqrt[3]{B} = \alpha - \beta i, \; (\alpha, \, \beta \text{ real}).$$

With this knowledge, we can find the value of the y roots in terms of α and β.

$$y_1 = 2\alpha,$$

$$y_2 = \tfrac{-1+i\sqrt{3}}{2}(\alpha + \beta i) + \tfrac{-1-i\sqrt{3}}{2}(\alpha - \beta i) = -\alpha - \beta\sqrt{3},$$

$$y_3 = \tfrac{-1-i\sqrt{3}}{2}(\alpha + \beta i) + \tfrac{-1+\sqrt{3}}{2}(\alpha - \beta i) = -\alpha + \beta\sqrt{3},$$

which shows that all our y roots are real. Now let's prove that they are distinct. Assume by way of contradiction that:

$$y_1 = y_2$$

$$\implies 2\alpha = -\alpha - \beta\sqrt{3}$$

$$\implies \beta = -\alpha\sqrt{3}.$$

This means that:

$$\sqrt[3]{A} = \alpha - \alpha i\sqrt{3}$$

$$\implies A = (\alpha - \alpha i\sqrt{3})^3 = \alpha^3 + 3\alpha^3 i\sqrt{3} - 3\alpha^3 i\sqrt{3} - 9\alpha^3 = -8\alpha^3,$$

which is contradictory to the fact that A is complex. So, $y_1 \neq y_2$, and using a similar argument, we can prove that $y_1 \neq y_3$. Moreover, it is clear that as $\beta \neq 0$, $y_2 \neq y_3$.

So, as a conclusion, all three of our roots are distinct and real.

Now let us try to express our roots in trigonometric form. This is convenient for further work to be done as our research progresses. In order to comfortably do this, we need to express A

and B in trigonometric form. Remembering that for the case when$\Delta < 0$ we have:

$$A = -\tfrac{q}{2} + i\sqrt{-\tfrac{q^2}{4} - \tfrac{p^3}{27}},$$

$$\text{we let } A = [\rho, \phi]$$

$$\implies \rho^2 = \tfrac{q^2}{4} - \tfrac{q^2}{4} - \tfrac{p^3}{27} = -\tfrac{p^3}{27}$$

$$\implies \rho = \tfrac{-p\sqrt{-p}}{27},$$

and

$$\implies \phi = \arccos(\tfrac{-q}{2\rho}) = \arccos(\tfrac{q\sqrt{27}}{2p\sqrt{-p}}).$$

This means that as $A = [\rho, \phi]$ and $B = A^* = [\rho, -\phi]$:

$$\sqrt[3]{A} = \sqrt{\tfrac{-p}{3}}(\cos\tfrac{\phi}{3} + i\sin\tfrac{\phi}{3}),$$

$$\sqrt[3]{B} = \sqrt{\tfrac{-p}{3}}(\cos\tfrac{\phi}{3} - i\sin\tfrac{\phi}{3}).$$

With the above, we are now ready to calculate the y roots in trigonometric form:

$$y_1 = \sqrt[3]{A} + \sqrt[3]{B} = 2\sqrt{\tfrac{-p}{3}}\cos\tfrac{\phi}{3},$$

$$y_2 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B} = [1, \tfrac{2\pi}{3}][\sqrt{\tfrac{-p}{3}}, \tfrac{\phi}{3}] + [1, -\tfrac{2\pi}{3}][\sqrt{\tfrac{-p}{3}}, -\tfrac{\phi}{3}]$$

$$= [\sqrt{\tfrac{-p}{3}}, \tfrac{2\pi}{3} + \tfrac{\phi}{3}] + [\sqrt{\tfrac{-p}{3}}, -\tfrac{2\pi}{3} - \tfrac{\phi}{3}] = 2\sqrt{\tfrac{-p}{3}}\cos(\tfrac{2\pi}{3} + \tfrac{\phi}{3}),$$

$$y_3 = \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B} = [1, -\tfrac{2\pi}{3}][\sqrt{\tfrac{-p}{3}}, \tfrac{\phi}{3}] + [1, \tfrac{2\pi}{3}][\sqrt{\tfrac{-p}{3}}, -\tfrac{\phi}{3}]$$

$$= [\sqrt{\tfrac{-p}{3}}, -\tfrac{2\pi}{3} + \tfrac{\phi}{3}] + [\sqrt{\tfrac{-p}{3}}, \tfrac{2\pi}{3} - \tfrac{\phi}{3}] = 2\sqrt{\tfrac{-p}{3}}\cos(\tfrac{2\pi}{3} - \tfrac{\phi}{3}).$$

3 Our Theorem

We are trying to prove the following theorem as the next major step of our research project:

If, for some real η, the three roots of a cubic are in the interval $[-2, 2+\eta]$, then they all are in the interval $[-2, 2]$.

At this point we suffice to say that different approaches have been taken to tackle this problem. Moreover, I am developing a C program that calculates all three roots of a cubic for a certain range of each of a, b, and c. The purpose of this program will be to give us an idea of what value we should choose for η. We note here that we are looking for the largest possible value of η under which our theorem will hold.

4 Conclusion

We conclude this midterm report by saying that this project so far has been an interesting experience, even though it has brought me personally a lot of frustration at many times. Moreover, it brought me down to Earth on many occasions. However, this will hopefully be a reminder that research is to a very large degree about persistence, and even though we might not get to the result we want at the speed we would like, the main point is to have a free thinking mind that is ready to look for the solution in all directions.