Proving the Lehmer Conjecture for Polynomials of Degree 10

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Given a polynomial $P(x_1, x_2, \ldots, x_k)$, the Mahler measure of P is defined as

$$M_k(P) \equiv \exp\Big[\int_0^1 \cdots \int_0^1 \ln|P(e^{2\pi\imath t_1}, \ldots, e^{2\pi\imath t_k})| dt_1 \cdots dt_k\Big] \tag{1}$$

which, for a polynomial of the form $P(x) = a\prod_{i=1}^{\epsilon}(x - \alpha_i)$, can be written as

$$M(P) = |a| \prod_{i=1}^{n} \max\{1, \alpha_i\} \tag{2}$$

Lehmer's conjecture states that there is a number $C > 1$ such that $M(f(x)) < C$ implies that $M(f(x)) = 1$. Although the Mahler measure is not well-understood, there are several theorems which are known. Among these are:

Theorem 1 $M(f(x)g(x)) = M(f(x))M(g(x))$

Theorem 2 $M(f(-x)) = M(f(x))$

Theorem 3 $M(x^n f(\frac{1}{x})) = M(f(x))$

Theorem 4 *If* $M(f(x)) = 1$*, then* $f(x)$ *is a factor of* $x^k - 1$ *for some* k*.*

Theorem 5 *If* $f(x) \neq x$ *is irreducible and not a factor of* $x^k - 1$ *for any* k*, then* $M(f(x)) < 1.4$ *implies that the coefficients of* $f(x)$ *are symmetric.*

In 1933, Lehmer stated that the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \tag{3}$$

has the lowest known non-unity Mahler measure of a polynomial of degree 10, which is $M = 1.176280821$. Since then, an equation of smaller non-unity Mahler measure has not been found. Thus, as it stands, $C = 1.176280821$.

Using the known theorems, the purpose of this summer research project is to prove Lehmer's conjecture for polynomials of degree 10. This will be accomplished by formally proving Lehmer's conjecture for small polynomials and extending the results to polynomials of larger degree. Althouth this has already been established computationally, the purpose will be to place on solid theoretical footing the validity of the conjecture for polynomials of small degree.