Penrose Tilings and Periodicity

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Project: This is the first part of a continuing project on non-commutative geometry. The space of all penrose tilings offers an interesting example of an NCG; work done this semester covers properties of tilings in general as well as specific properties of penrose tilings. This project is supported by a UofA undergraduate research assistantship. Work done was assisted and supervised by Arlo Caine, a graduate student, and his thesis advisor, Dr. Doug Pickrell.

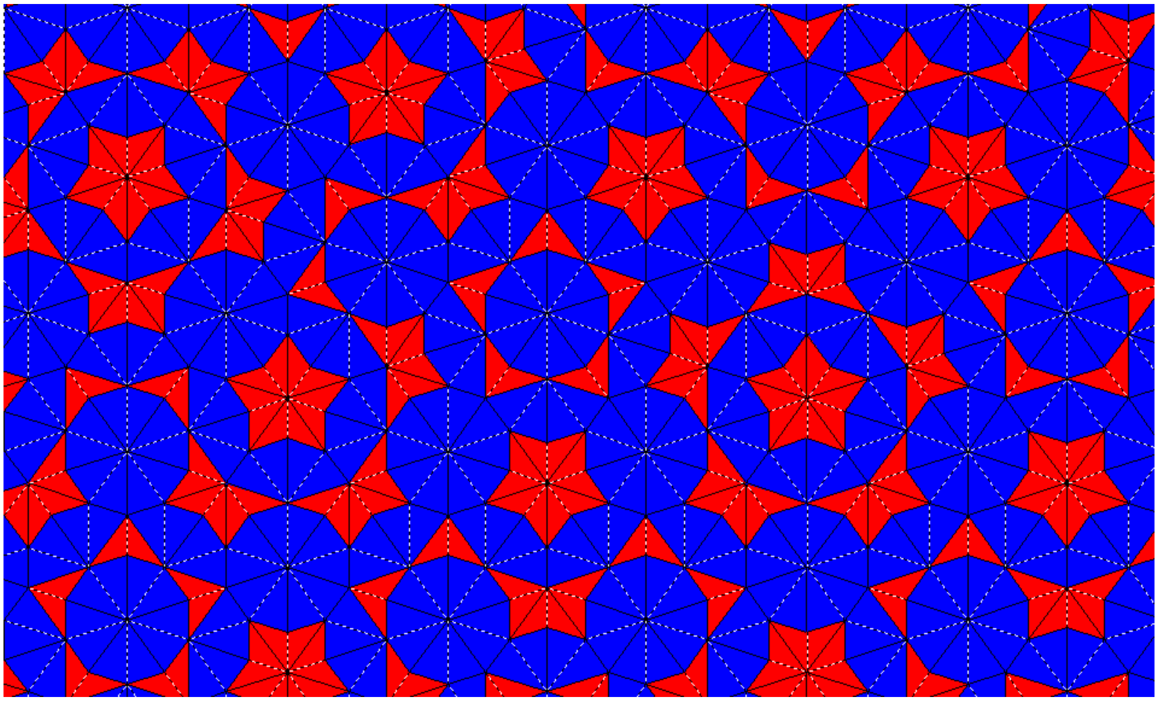
We begin our discussion of Penrose Tilings with a discussion of the more general concept of tilings. Two-dimensional tilings are a plane filling arrangement of a finite number of prototiles, such that the entire plane is covered with a prototile and no tiles overlap. The most familiar example might be an infinite grid of squares. In this case, the prototile is the square. This is also one of the relatively few tilings that can be constructed using identical, regular n-gons. In fact, the only regular n-gons to tile the plane are the equilateral triangle, the square, and the hexagon.

Proof: Consider a tiling of the plane by regular n-gons for some $n \in \mathbb{N}$. A tiling induces a graph on the plane. Thus, at any particular vertex, there is some $k \in \mathbb{N}$ such that $k \cdot (\angle_{\text{int}}) = 360^\circ$. It is well known that the interior angle of a regular n-gon is given by $\angle_{\text{int}} = \frac{(n-2)\cdot 180^\circ}{n}$. Thus, $\frac{k\cdot(n-2)\cdot 180^\circ}{n} = 360$, so $\frac{2n}{n-2} = k$. Let $I = n-2$. Then $n = I+2$, so $\frac{2(I+2)}{I} = k \in \mathbb{N}$. Thus, $2 + \frac{4}{I} = k$. Since $k \in \mathbb{N}$, $k-2 \in \mathbb{Z}$ so $\frac{4}{I} = k-2 \in \mathbb{Z}$. If $\frac{4}{I} \in \mathbb{N}$, then I is a factor of 4, i.e. I$\in$ {-4,-2,-1,1,2,4}. Since $n = I+2$, and $n \in \mathbb{N}$, $n \in \{3,4,6\}$. Thus, the set of regular n-gons satisfying the necessary conditions for an edge-to-edge tiling are the regular 3-gon, 4-gon, and 6-gon, or the equilateral triangle, square, and regular hexagon.

It turns out that generalizing the conditions for whether prototiles can tile the plane or not is much more complicated as the number of prototiles increases. As an example, we have found that two different prototiles, a regular n-gon and 2n-gon, can only tile the plane if the figures considered are the pentagon and decagon, the square and octagon, or the triangle and hexagon. Note the increased complexity of the proof, found in the appendix. Considering more general polygons, and more general numbers of prototiles would require different methods of analysis.

Periodicity. An important distinction to make between tilings of the plane is whether they exhibit translational symmetry. For example, the tiling consisting of a grid of squares of unit length is periodic because translation up "a" units and across "b" units yields the same pattern $\forall a,b \in \mathbb{Z}$. Choose a vertex. Then, the vector $\langle \mathrm{a}, \mathrm{b} \rangle$ beginning at this vertex is a vector of translation symmetry. It is easy to think of other, more complex tilings, which also exhibit translational symmetry. Consider the tiling of regular decagons and regular pentagons. A vector between the center of any pentagon and the center of any other pentagon describes translational symmetry.

As one constructs tilings, it seems difficult to conceive of constructing aperiodic tilings of the plane. Penrose Tilings, as will be discussed later, are examples of such aperiodic tilings of the plane.



Penrose Tilings. Prototiles are fundamental shapes which can be arranged on a plane to form a tiling. Penrose tilings consist of two isosceles triangles, one with an acute vertex angle and one with an obtuse vertex angle, such that the diagram in Figure 1 is valid. These prototiles have an interesting "scaling" relationship that one can think of as analogous to the golden rectangle. The following proof is a construction of a double decomposition, and demonstrates the relationship of the successive sizes of prototiles.

The fundamental tiles of a Penrose Tiling decompose twice with the ratio $1:\tau$.

Proof:

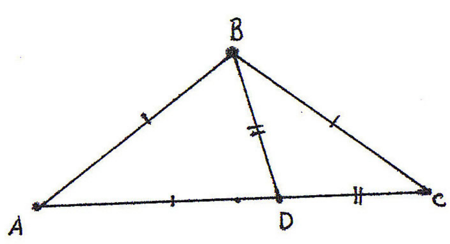


Figure 1

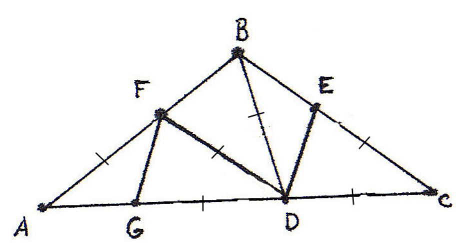


Figure 2

We assume Figure 1. We then construct a line from D to some point E on $\overleftrightarrow{BC}$ such that $\angle EDC = \angle BDA$. Since $\triangle ABC$ is isosceles, $\angle BAC = \angle BCA$. Thus, $\triangle CED \sim \triangle ABD$ by AA similarity. Note that $\triangle BDC$ is isosceles, so that $\angle ACB = \angle CBD$. Call this angle α. Then, considering $\triangle ABC$, and letting

$\angle ABD = \beta$, we find that $3\alpha + \beta = \pi$. Now consider $\triangle CDE$. It must also be isosceles since it is similar with an isosceles triangle. Thus, we find from $\triangle DEC$ that $\alpha + 2\beta = \pi$. Combining these expressions, $\alpha = \frac{\pi}{5}$ and $\beta = \frac{2\pi}{5}$. We then construct a line from D to some point F on $\overleftrightarrow{AB}$ such that $\angle FDA = \alpha$. Then, $\angle FDA = \angle FDB$. Now construct a line from F to some point G on $\overleftrightarrow{AD}$ such that $\angle FGD = \beta$. Clearly, $\triangle GDE \cong \triangle FDB$ by AAS congruence. It is also clear that these triangles are congruent to $\triangle DCE$ by AAS. Consider $\triangle DCB$. Since it is isosceles, $\angle DCB = \angle DBC$ and by transitivity, we have that $\triangle DBC \sim \triangle ABC$ by AA congruence. Thus, $\frac{|DC|}{|BC|} = \frac{|BC|}{|AC|}$. But, $|AC| = |AD| + |DC|$ and $|AD| = |BC|$. Thus, $\frac{|DC|}{|BC|} = \frac{|BC|}{|BC| + |DC|}$. Arbitrarily, let $|DC| = 1, |BC| = x$. Then $x^2 = x + 1 \Rightarrow$ $x^2 - x - 1 = 0 \Rightarrow x = \frac{1+\sqrt{5}}{2} = \tau$. It is a simple exercise to finish and show that $\triangle AGF \cong \triangle DEB$ and that the ratio of similarity between $\triangle AGF, \triangle CDB$ is τ.

Infinite Plane. This decomposition process is perhaps the most fundamentally important aspect of Penrose Tilings, because it is what insures that Penrose Tilings exist, which are 'arbitrarily large.' By successively repeating this double decomposition to each prototile, and then enlarging the dimensions of the graph to retain the prototiles original size, essentially we have expanded the graph by a factor of τ. In this limiting process, we can ask another important question. What is the ratio of tiles with an acute vertex angle, to tiles with an obtuse vertex angle? We find that the ratio is τ!

Proof:

Let λ_N be the ratio of prototiles with acute vertex angles to those with obtuse angles after N double decompositions. We note that after every double decomposition, the obtuse prototiles each split into two new obtuse prototiles and an acute prototile, while each original acute prototile splits into a new obtuse prototile and a new acute one. In symbols representing the number of obtuse and acute prototiles at each stage N of double decomposition, $O_{N+1} = 2O_N + A_N, A_{N+1} = O_N + P_N$. Thus,

$$\lambda_{N+1} = \frac{O_{N+1}}{A_{N+1}} = \frac{2\frac{O_N}{A_N} + 1}{\frac{O_N}{A_N} + 1} = \frac{2\lambda_N + 1}{\lambda_N + 1}.$$

What is the limit of λ as $N \to \infty$? Consider the Fibonacci numbers defined by the sequence $U_N = U_{N-1} + U_{N-2}$. Define the function $\mathbb{F}(\mathrm{n}) : \mathbb{N} \to \mathbb{N}$ to be the function such that $\mathbb{F}(0) = 1, \mathbb{F}(1) = 1$, and $\forall n > 1, \mathbb{F}(n) = \mathbb{F}(n-1) + \mathbb{F}(n-2)$.

Then the value $\lambda_N = \frac{\left(2\mathbb{F}(2N)+\mathbb{F}(2N-1)\right)\lambda_0 + \left(2\mathbb{F}(2N)+\mathbb{F}(2N-1)\right)}{\left(2\mathbb{F}(2N-1)+\mathbb{F}(2N-2)\right)\lambda_0 + \left(2\mathbb{F}(2N-1)+\mathbb{F}(2N-2)\right)}$, as can be shown inductively. Since $\mathbb{F}$ is an increasing function, it is easily seen that $\lim_{N\to\infty}(\lambda_N) = \lim_{N\to\infty}\left(\frac{2\mathbb{F}(2N)+\mathbb{F}(2N-1)}{2\mathbb{F}(2N-1)+\mathbb{F}(2N-2)}\right)$. But by properties of $\mathbb{F}$, this reduces to $\lim_{N\to\infty}\left(\frac{\mathbb{F}(2N)+\mathbb{F}(2N+1)}{\mathbb{F}(2N-1)+\mathbb{F}(2N)}\right) = \lim_{N\to\infty}\left(\frac{\mathbb{F}(2N+2)}{\mathbb{F}(2N+1)}\right) = \lim_{a\to\infty}\left(\frac{\mathbb{F}(a+1)}{\mathbb{F}(a)}\right)$. Since $\mathbb{F}(a+1) = \mathbb{F}(a)+\mathbb{F}(a-1)$, $\frac{\mathbb{F}(a+1)}{\mathbb{F}(a)} = 1 + \frac{\mathbb{F}(a-1)}{\mathbb{F}(a)}$. The $\lim_{a\to\infty}\left(\frac{\mathbb{F}(a+1)}{\mathbb{F}(a)}\right)$ exists, so let $\lim_{a\to\infty}\left(\frac{\mathbb{F}(a+1)}{\mathbb{F}(a)}\right) = L$. Then $\lim_{a\to\infty}\left(\frac{\mathbb{F}(a-1)}{\mathbb{F}(a)}\right) = \frac{1}{L}$. So we find that $L = 1 + \frac{1}{L} \Rightarrow L^2 = L+1 \Rightarrow L = \tau$. Thus, $\lim_{N\to\infty}(\lambda_N) = \tau$.

Aperiodicity. The following argument is presented in *Tilings, Chaotic Dynamical Systems and Algebraic K-Theory* (see Resources) and is paraphrased below.

> We now consider why Penrose Tilings are aperiodic. Suppose to the contrary that a Penrose tiling **T** was periodic, then there would be some vector **v** of translational symmetry. Let $\mathbf{T}'$ be the tiling after a process of deflation (where edges of the same type are erased, and the tiling is scaled down by a factor of $\frac{1}{\tau}$, leaving the prototiles the same size as the original tiling). Then $\mathbf{T}'$ still has the same symmetry vector **v**, though it has been reduced by $\frac{1}{\tau}$. Repeating this process sufficiently many times leads to a contradiction, because the vector **v** cannot be smaller than the size of the prototiles. Thus, no symmetry vector **v** is permissible, and the tiling is aperiodic.

Unfortunately, this argument is insufficient because it applies to periodic tilings that have decomposition. Consider the periodic tiling of equilateral triangles and regular hexagons. There exists a decomposition that erases edges and preserves sufficiently large symmetry vectors. However, given any vector of symmetry, the decomposition could be performed enough times to make the vector smaller than the length of either prototile.

A more rigorous argument for the aperiodicity of Penrose Tilings is required that utilizes the properties of Penrose Tilings that are unique from those of periodic tilings.

Appendix

Which two regular polygons of the form n and 2n will tile the plain edge-to-edge?

Pentagon and Decagon, Triangle and Hexagon, and Square and Octagon.

Proof: Since this is a tiling of two prototiles, there must exist at least one vertex where at least one of each prototile is present. We model the relationship of this vertex.

$$\exists k_1, k_2, n \in \mathbb{N} \text{ such that } \quad \frac{k_1(2n-2)180^\circ}{n} + \frac{k_2(n-2)180^\circ}{n} = 360^\circ \tag{1}$$

The first term in (1) is the contribution from the 2n-gon(s) and the second term is the contribution from the n-gon(s). Now, we seek to limit the values of k_1, k_2. Since the first term in (1) is going to be a positive number, we have

$$\frac{k_2(n-2)180^\circ}{n} < 360^\circ \qquad \Rightarrow \qquad 0 < k_2 < \frac{2n}{n-2}.$$

As noted in a previous result, $K = \frac{2n}{n-2} \in \mathbb{N} \Rightarrow K \in \{3,4,6\}$. Since $k_1 < K$, $k_1 \in \{1,2,3,4,5\}$. A similar argument limits k_2, and we find that $k_2 \in \{1,2\}$. We return to (1), and rewrite as $\exists k_1 \in \{1,2\} \ni \exists k_2 \in \{1,2,3,4,5\}$ such that for n>3,

$$\frac{k_1(2n-2)}{4n} + \frac{k_2(n-2)}{2n} = 1$$

This equation can be re-written to solve for n in terms of k_1, k_2.

$$n = \frac{k_1 + 2k_2}{k_1 + k_2 - 2}.$$

We hold k_1 fixed to determine what values of k_2 make the expression integral. An exhaustive check of the possible values of k_2 yields these results:

$k_1=1$: $\quad n = \frac{1+2k_2}{k_2 - 1} \in \mathbb{N} \quad \Leftarrow \quad k_2 \in \{2,4\}$

$k_1=2$: $\quad n = \frac{2+2k_2}{k_2} \in \mathbb{N} \quad \Leftarrow \quad k_2 \in \{1,2\}$

Note that we are only interested in the cases when n>2. Then the final set of (k_1, n_1, k_2, n_2) that satisfies equation (1) is

$$\{(1,10,2,5),(2,8,1,4),(1,6,4,3),(2,6,2,3)\}$$

This corresponds to the pentagon and decagon, the square and octagon, and two references to the equilateral triangle and hexagon. Any edge-to-edge tiling of regular n and 2n gons must be a subset of this set.

Resources

Tasnadi, Tamas. *Penrose Tilings, Chaotic Dynamical Systems and Algebraic K-Theory.* April 09, 2002. <http://arxiv.org/PS_cache/math-ph/pdf/0204/0204022.pdf>

B. Grunbaum and G.C. Shephard. *Tilings and Patterns.* Freeman, New York, 1989.