Undergraduate Research Project

Department of Mathematics
The University of Arizona

Dynamic Systems and Chaos

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Final Report

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Chaos Theory

Chaos is a fundamental property that possesses nonlinearity and a sensitive dependence on initial conditions. Because of the nonlinearity in a chaotic system it becomes very difficult to make accurate predictions about the system over a given time interval. Weather forecasting is an example of how chaos theory effects the accuracy of predictions over a given time interval. Through analyzing a weather pattern over time, meteorologists have been able to make better predictions of future weather based on this theory. Another area in the domain of chaos is the behavior of lasers. Specifically, the emission of a laser is affected by chaos due to feedback in the system. Feedback originates from the reflections in the optical cavity of the laser, and is amplified through multiple reflections and emissions. This feedback becomes chaotic as it leaves the optical cavity and enters the external cavity of the laser where a time delay takes place. Hence, the intensity of the emitted beam may be modeled by the chaotic properties of the external feedback. Therefore, in order to make predictions about a laser's intensity over a amount given time, chaos theory should be studied and applied to laser rate equations. [1]

Stability is an important concept when studying systems over a given time interval. A system is considered stable when a condition converges toward a single point within a set range. On the other hand, a system becomes unstable when conditions diverge from a fixed point and depart from this range. Further, when the system diverges and splits, creating a more complicated system. The locations of these splits are called bifurcation points. As the system progresses with time it exponentially develops more bifurcation points. These special points can be related to the chaotic behavior of two synchronized laser systems. This phenomenon of bifurcation points from synchronization may be modeled by Quadratic Maps. In order to find predictions in the system a connection must be formed between the initial conditions and the stability of the system.

Synchronization

The main goal of the current research on the chaotic properties of lasers is to establish the necessary conditions for synchronizing coupled lasers. The coupling of the lasers combines two individually chaotic systems into one system with a superposition of emissions. In order to analyze a chaotic coupled laser system, quadratic maps with iterations will be created and analyzed using semiconductor rate equations, which is illustrated in Fig 1.1 below.

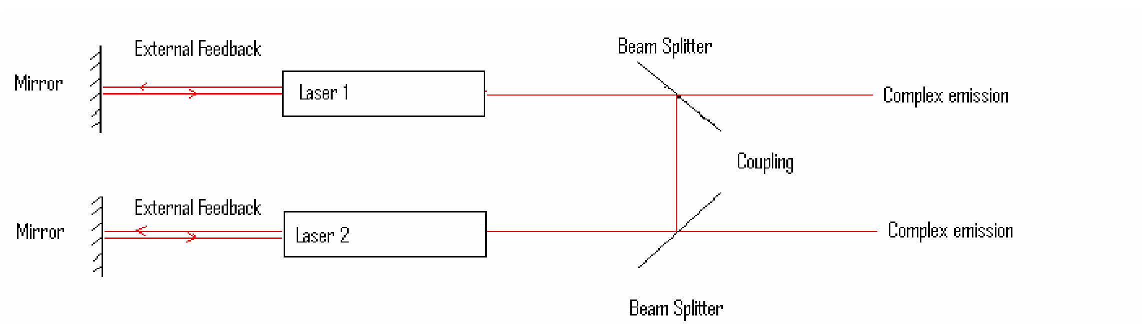


Figure 1.1 A model of two lasers being coupled by reflecting mirrors and beam splitters.

The Semiconductor Laser

Lasers have redefined the scientific world, commerce, and everyday living over the last century. Lasers may be found anywhere from scientific laboratories to supermarkets; and therefore, should be further explored and researched in order to progress as a technological society. Solid state, semiconductor, and gas lasers are just a few of the many different types of lasers available on the market today. Each of these different types of lasers are important for different applications based upon the desired result and cost. This research project examines the properties of semiconductor lasers because of their versatility and cost-efficiency.

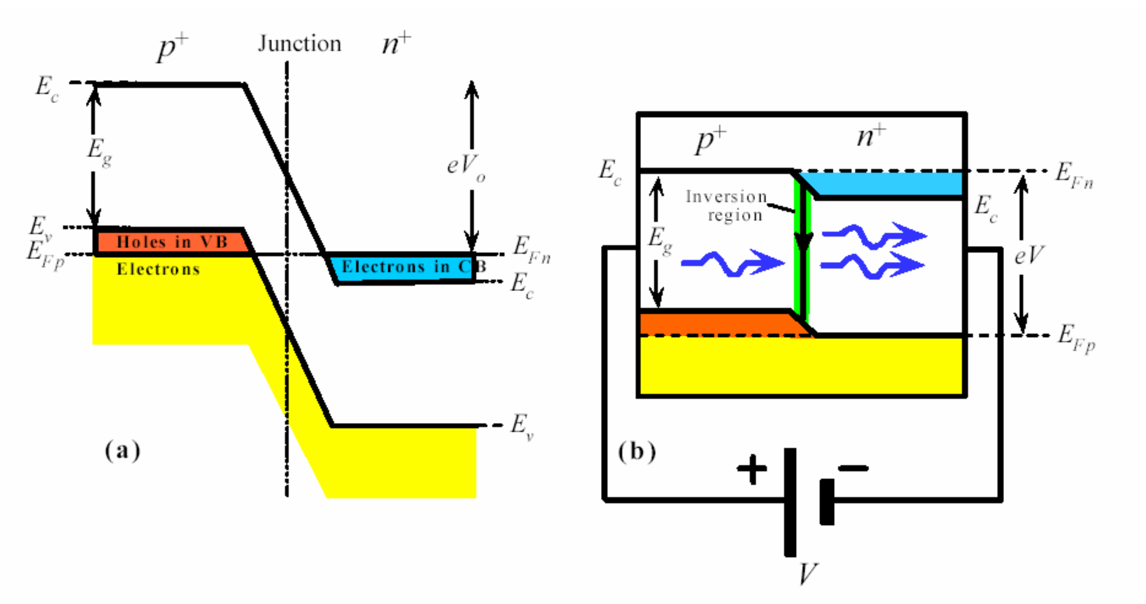
First, it is important to understand what a laser is and why it is a desirable light emitting source. Laser is an acronym for Light Amplification by Stimulated Emission of Radiation. As such, a laser amplifies light in order to create a desired emission. Light is amplified through the properties of p-n junctions, electron-hole pairs, optical gain, and population inversion. Here, the chaotic characteristics of laser emission will be explored. Understanding the chaotic nature of the laser will lead to accurate predictions of the emitted irradiance of the laser, which is necessary when studying more complex systems with feedback and coupling.

How a Laser Emits Light

A semiconductor laser is made up of several different components which lead to the production of optical gain; whereby yielding stimulated emission. The basic components of the laser consist of a current source, a semi-conductive material, and an optical cavity. Each of these components controls the stimulated emission of the laser and need to be understood in order to perform well-founded computational analysis.

Semi-conductive Material

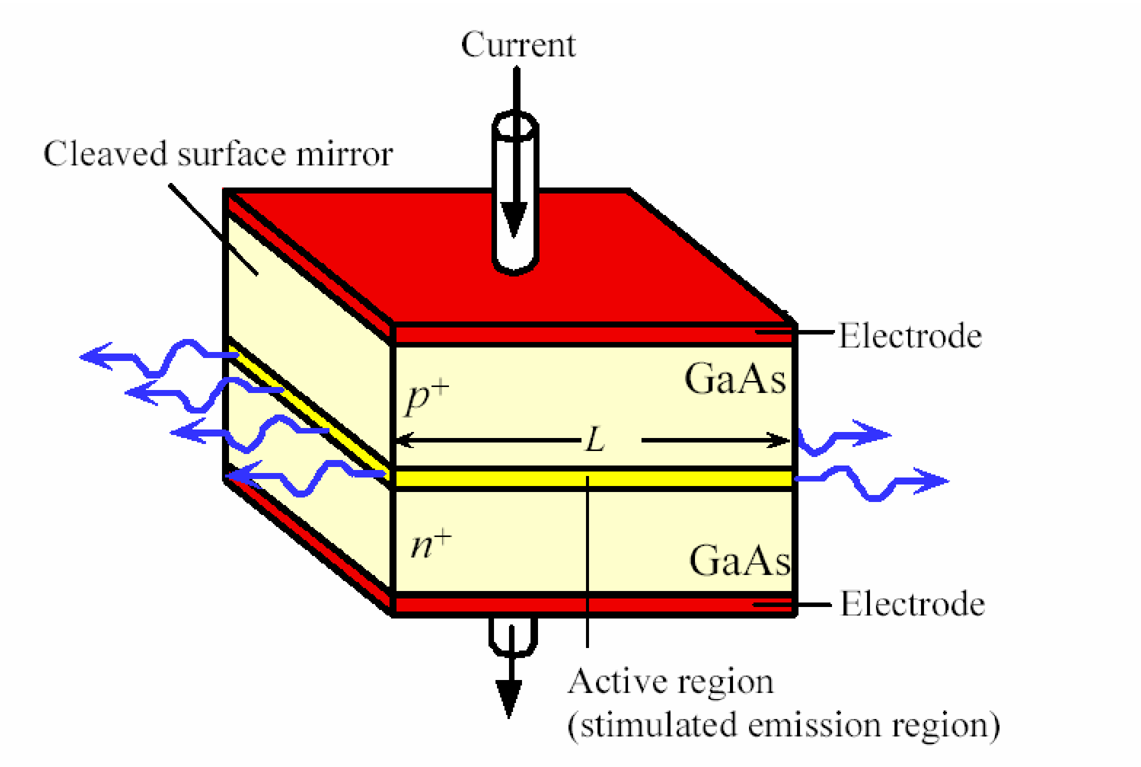
A material is considered semiconductive if it has conductance, but its conductance must be less than that of a metal. The semiconductive material used in a semiconductor laser is composed of electron-hole pairs defining the n-type and p-type junctions of the material. For a material to be either n- or p-type an impurity must be inserted into the crystal lattice. This process is termed 'doping'. If a larger amount of electrons are inserted in the lattice of the material, it becomes n-type. In contrast, if the doping introduces a larger amount of holes, then the lattice is considered to be p-type. These properties allow for stimulated emission to occur through using a current source to create pumping. [2] [3]



Population Inversion

In order for the laser to emit light population inversion must be present in the system. Population inversion occurs when the initial carrier density is less than the final carrier density, and when the initial energy is less than the final energy. This process occurs from the consequences of pumping in the optical cavity. [2]

An optical cavity is necessary to produce a stimulated emission through pumping. This cavity is a region composed of two approximately parallel mirrors separated by a defined distance. The semiconductive material used in the laser forms the optical cavity in the case of the semiconductor laser. Moreover, the parallel edges of the material have mirror like properties themselves and are separated by the length of the material in order to form the optical cavity, which then produces a positive feedback. [2]



Gain is a measurement that is determined by the length of the optical cavity and the number of reflected passes though the semiconductive material. For each pass through the optical cavity a loss occurs due to the mirrors that is proportional to the gain. When pumping is applied to the semiconductive material, the gain increases for each pass through the optical cavity. Population inversion occurs when the gain reaches a value higher than the loss from reflection. [2]

Semiconductor Laser Rate Equations

The basic operation of the semiconductor laser may be described by the following equations:

$$\frac{dE}{dt} = -\alpha_i E + k((N-1) - i\alpha N)E \qquad (1.1)$$

$$\frac{dN}{dt} = J - \gamma N - 2k(N-1)|E|^2 \qquad (1.2)$$

It is important to understand the change in the complex amplitude and carrier density of the laser since these fundamental factors allow for population inversion to occur, leading to emission. [4]

Component	**Description**
E	Electric field envelope (Complex amplitude)
N	Carrier density
$-\alpha_i * E$	Loss
$-i\alpha N$	Change to refractive index (dependent on carrier density)
J	Pumping (current)
$-\gamma N$	Loss of carriers from spontaneous recombination
$k\,[(N-1)]E$	Gain from stimulated emission
$-2k(N-1)$ $abs(E)^2$	Consumption of carriers from stimulated emission

Table 1. Description of variables

Stability and Fixed Points

Stability of the *Intensity System*

To begin, we shall analyze the stability of what is to be called, the Intensity System.

The laser system will be modeled as a system coupled differential equations, as previously studied by [1], which relates the complex amplitude (*E*) and carrier density (*N*) as functions of time by:

$$\frac{dE}{dt} = -\alpha_i E + k((N-1) - i\alpha N)E \qquad (1.1)$$

$$\frac{dN}{dt} = J - \gamma N - 2k(N-1)|E|^2 \qquad (1.2)$$

Equations (1.1) and (1.2) form a map which characterizes the behavior of the semiconductor over time [1]. This set of equations shall be termed the *Complex Amplitude System*. As will be explained, we seek a steady point on this map, which satisfies:

$$\frac{dE}{dt} = 0 = \frac{dN}{dt} \qquad (1.3)$$

To begin this analysis, (1.1) is transformed into a differential equation for intensity (*I*) by using the relation: $I = |E|^2 = E^*E$:

$$\begin{aligned}\frac{dI}{dt} &= \frac{d(E^* E)}{dt} \\ &= E^* \frac{dE}{dt} + E\frac{dE^*}{dt} \\ &= E^* E(-\alpha_i - \alpha_i + k((N-1) - i\alpha N) + k((N-1) + i\alpha N)) \\ &= 2I(-\alpha_i + k(N-1))\end{aligned}$$

Since (1.2) already contains the intensity term *I* it will not require further manipulation. With this new equation the *Intensity System* is established:

$$\frac{dI}{dt} = 2I[-\alpha_i + k(N-1)] \qquad (1.4)$$

$$\frac{dN}{dt} = J - \gamma N - 2k(N-1)I \qquad (1.5)$$

As such, we seek a special point where the system displays equilibrium, namely a fixed point. Physically, our fixed point of interest is when the laser is emitting a coherent beam of light [2]. In this case the complex amplitude may be said to be constant with respect to time, and for a fixed point this condition further implies the same form of (1.3):

$$\frac{dI}{dt} = 0 = \frac{dN}{dt} \qquad (1.6)$$

In solving (1.6), (1.4) takes the form:

$$\alpha_i\, I = k(N-1)\, I \Rightarrow I = 0\,,\; N = \frac{\alpha_i}{k} + 1$$

The former solution suggests that the laser is off, which is not of particular interest. Therefore, to retrieve the corresponding "on" fixed point for the intensity (*I*) the latter is chosen and used in the zero solution of (1.5):

$$\begin{aligned} &J = \gamma N + (N-1)\, I \\ &\Rightarrow J = \gamma\left(\frac{\alpha_i}{k} + 1\right) + 2\,k\left(\frac{\alpha_i}{k} + 1 - 1\right) I \\ &\Rightarrow J - \gamma\left(\frac{\alpha_i}{k} + 1\right) = 2\,\alpha_i\, I \\ &\Rightarrow I = \frac{1}{2}\left(\frac{J-\gamma}{\alpha_i} - \frac{\gamma}{k}\right) \end{aligned}$$

In summary, the (on) fixed point of interest is:

$$(I_o, N_o) = \left(\frac{1}{2}\left(\frac{J-\gamma}{\alpha_i} - \frac{\gamma}{k}\right),\; \frac{\alpha_i}{k} + 1\right)$$

Since a fixed point may be defined as point that possesses attractors about an orbit, it follows that small deviations from this orbit will be stable if they return to the fixed point. Further, these deviations may be regarded as perturbations on our system of differential equations [1]. In order to test the stability of this fixed point a linearization is performed with perturbations variables χ and η on *Io* and *No,* respectively. Under these stipulations we have that:

$$I_o \to I_o + \chi,\; N_o \to N_o + \eta, \quad \ni \chi, \eta \ll 1, \chi \in \mathbb{C},\; \eta \in \mathbb{R}$$

Inserting these new values of I_o and N_o into the *Intensity System,* canceling zeroth order terms, i.e. **parent solutions** and doubly small terms ($\chi\eta$). These steps in linearization come from the fact that the behavior of the **parent solution** is known and the zeroth order terms are negligibly small deviations as compared to the remaining terms. Performing the linearization yields :

$$\frac{d(I_o+\chi)}{\mathrm{dt}} = 2\,(I_o + \chi)\,(-\alpha_i + k(N_o + \eta - 1)) =$$

$$\frac{\mathrm{d}\chi}{\mathrm{dt}} = 2\,I_o\,(-\boldsymbol{\alpha_i} + \boldsymbol{k}(\boldsymbol{N_0} - \mathbf{1})) + 2\,\chi(-\alpha_i + \boldsymbol{k}(\boldsymbol{N_o} - \mathbf{1} + \eta)) + 2\,I_o\,k\eta$$

$$= -2\,\chi\alpha_i + 2\,k\chi(N_o + \eta - 1) + 2\,I_o\,k\eta$$

$$= 2\,(k\,(N_o - 1) - \alpha_i)\,\chi + 2\,(I_o\,k)\,\eta$$

Similarly,

$$\frac{d(N_o+\eta)}{\mathrm{dt}} = J - \gamma(N_o + \eta) - (N_o + \eta - 1)\,(I_o + \chi) =$$

$$\frac{\mathrm{d}\eta}{\mathrm{dt}} = \boldsymbol{J} - \boldsymbol{\gamma N_o} - \boldsymbol{I_o}(\boldsymbol{N_o} - \mathbf{1}) - \gamma\eta - N_o\,\chi - I_o\,\eta + \chi$$

$$= (1 - N_o)\,\chi - (\gamma + I_o)\,\eta$$

Hence, the perturbed *Intensity System* under linearization is:

$$\frac{\mathrm{d}\chi}{\mathrm{dt}} = (\,2\,(k(N_o - 1) - \alpha_i))\,\chi + (2\,I_o\,k)\,\eta$$

$$\frac{\mathrm{d}\eta}{\mathrm{dt}} = (1 - N_o)\,\chi - (\gamma + I_o)\,\eta$$

The above system is a linear system of differential equations, which may be rewritten in matrix form as:

$$\frac{d}{\mathrm{dt}}\begin{pmatrix}\chi\\ \eta\end{pmatrix} = \begin{pmatrix}w & x\\ y & z\end{pmatrix}\begin{pmatrix}\chi\\ \eta\end{pmatrix}$$

Where $w = 2\,(-\alpha_i + k(N_o - 1))$,
$x = 2\,I_o\,k,\;\; y = 1 - N_{o,},$ and $z = -\gamma - I_o$

Further, denote this coefficient matrix as the matrix **A**. By induction, this perturbed system has a solution in the form:

$$\begin{pmatrix}\chi\\ \eta\end{pmatrix}[t] = \boldsymbol{C}\,e^{\mathbf{A}t} \qquad (1.8)$$

where the matrix **C** is a 2x1 matrix with initial conditions $\chi(0)$ and $\eta(0)$ as its elements. The exponential matrix formed by $e^{\mathbf{A}}$ is established as a valid solution since **A** is full of rank. Therefore, it possesses two linearly independent eigenvectors ($[\lambda_1,\ \lambda_2] \in \mathbf{P}$) such that it may be reduced to diagonal form via a similarity transformation: $\mathbf{A} = \mathbf{BPB}^{-1}$. With these conditions the the matrix exponential may be defined by using the Maclaurian Series:

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A}\,\frac{1}{2!}\,\mathbf{A}^2 + \frac{1}{3!}\,\mathbf{A}^3 + \ldots + \frac{1}{n!}\,\mathbf{A}^n = \mathbf{PD\,P}^{-1}$$

where **D** is diagonal with elements e^{λ_i} [5]. Finally, it is established that if the real parts of the eigenvalues of **A** are less than zero, then the "on" fixed point for the *Intensity System* will be stable. To find these eigenvalues the typical method of setting det(**A - λI)** equal to zero leads to:

$$\lambda = \frac{-(\gamma + I_o) \pm \sqrt{(\gamma + I_o)^2 + 8\,(I_o(1 - N_o))}}{2}$$

Analyzing the above equation shows that the critical points occur at $I_o = 0$ and $N_o = 1$. Relating these conditions to (1.7) yield that the *Intensity System* will be stable so long as the current (J) is less than $\gamma(\frac{\alpha_i}{k} + 1)$ and that $\frac{\alpha_i}{k}$ be greater than zero. These implications are physically sound since the current is easily controlled in the laboratory setting and the loss coefficient is usually greater than its gain term. To verify these conclusions, *Matlab* was used to produce the following graphs:

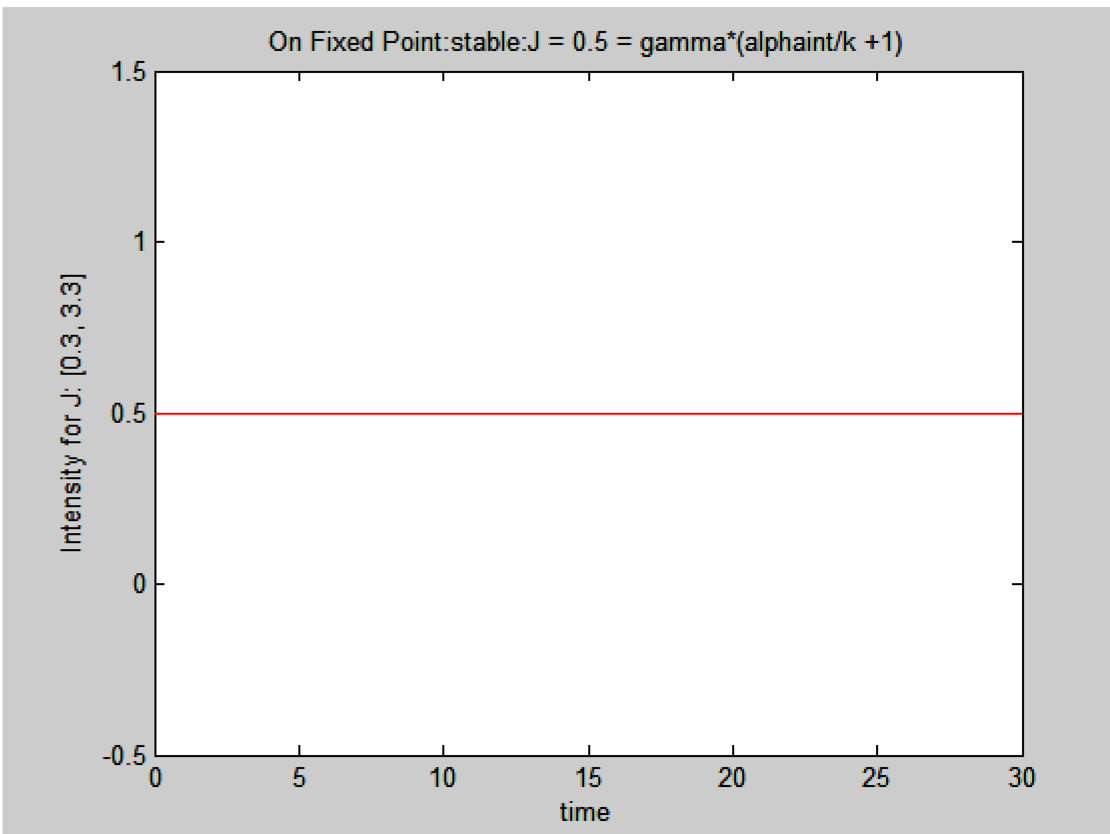


Figure 2.1 *The above graph shows that for the current (J) remains stable at the critical value of the scaled gain and gamma terms.*

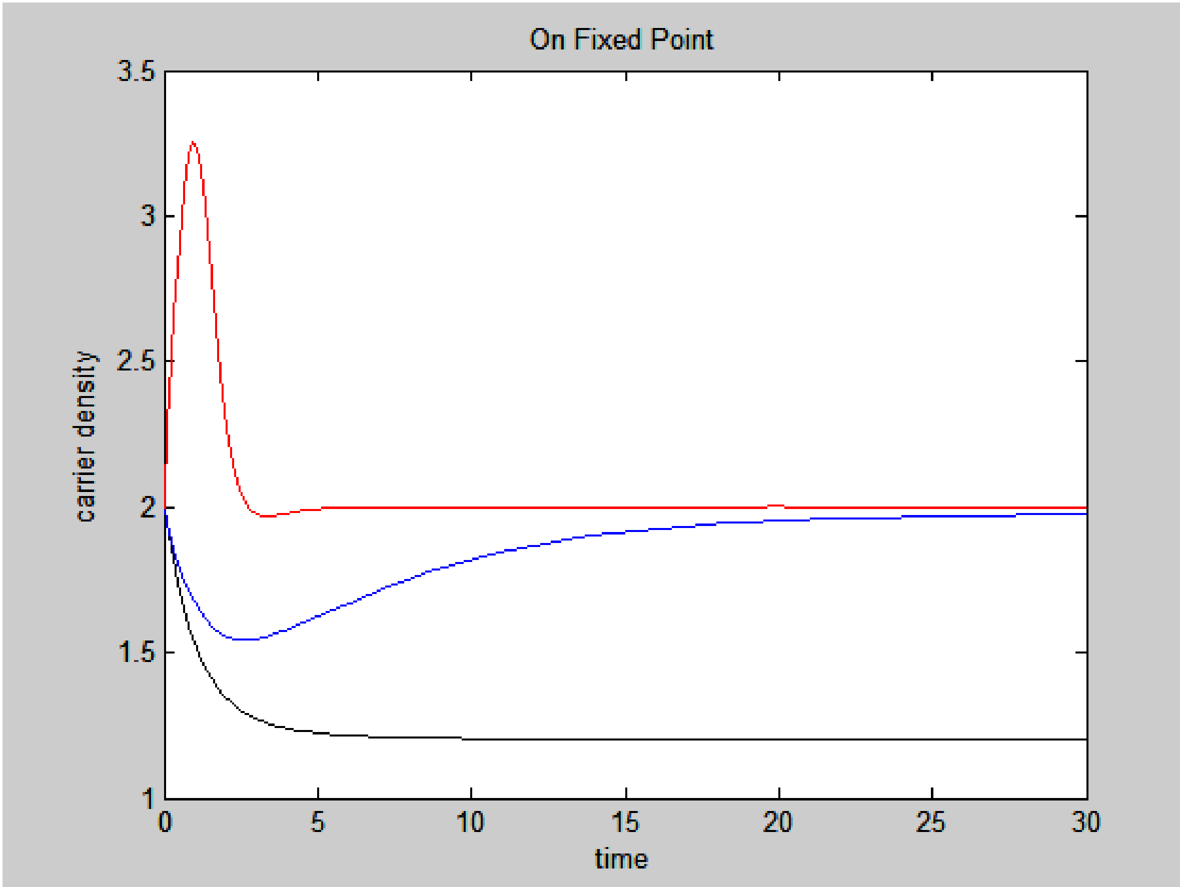


Figure 2.2 *As expected from the analysis, carrier density becomes stable after a short amount of time.*

Next, the Complex Amplitude System's stability behavior is analyzed.

Stability of the *Complex Amplitude System*

It is important to observe the stability of the complex amplitude in order to further develop the relationship of feedback concerning the electric field envelope and the carrier density. Since the complex amplitude is exponential the solution for the phase must be negative in order to be stable based on convergence. Therefore, the stability of the complex amplitude is found through solving for eigenvalue solutions and observing sign conventions for real solutions. The first step in analyzing the stability is to form the complex amplitude equation:

$$E_o = A_o e^{-i\alpha N_o t} \qquad (2.1)$$

Where A_o is the amplitude and $-\alpha N_o t$ is the phase.

In the same sense that the *Intensity System's* stability is studied for the "on" case, the *Complex Amplitude System's* "on" fixed point is examined. Starting from (2.1), the amplitude is extracted from the previous fixed point:

$$(I_o, N_o) = \left(\frac{1}{2}\left(\frac{J-\gamma}{\alpha_i} - \frac{\gamma}{k}\right),\ \frac{\alpha_i}{k} + 1\right) \qquad (2.2)$$

(2.1) becomes complete upon using the aforementioned relationship between amplitude and intensity:

$$A_o = \sqrt{I_o} = \sqrt{\frac{1}{2}\left(\frac{J-\gamma}{\alpha_i} - \frac{\gamma}{k}\right)} \qquad (2.3)$$

Linearization will now be implemented to the primary equations (2.1) and (2.2) for stability analysis similar to the preceding work. The perturbation variable for the complex amplitude will be defined as *e*, and the perturbation variable for the carrier density will be defined as *n*.

$$E_o \to E_o + e,\ N_o \to N_o + n,\ \ni e,\ n \ll 1, e \in \mathbb{C},\ n \in \mathbb{R}$$

Since the absolute value of (2.3) is complex, a new term is substituted into (1.1):

$$|E_o + e|^2 = (E_o + e)(E_o^* + e^*)$$

To solve for the eigenvalues of this perturbed Complex Amplitude system, a 3x3 matrix is formed using (1.1) and (1.2). After substitution and simplification, a system of three linear equations is developed:

$$\frac{dE}{dt} = -i\,k\,\alpha\,N_o\,e + k\,E_o(1 - i\alpha_i)\,n \qquad (2.6)$$

$$\frac{de^*}{dt} = -i\,k\,\alpha\,N_o\,e^* + k\,E_o^*(1 - i\,\alpha) \qquad (2.7)$$

$$\frac{dn}{dt} = -k\,E_o^*(N_o - 1)\,e - 2\,k\,E_o\,(N_o - 1)\,e^* - (2\,k\,|A_o|^2 + \gamma)\,n \qquad (2.8)$$

The following matrix is used to solve the eigenvalues of the complex amplitude and is dependent upon e, e^*, and n :

$$\frac{d}{dt}\begin{pmatrix} e \\ e^* \\ n \end{pmatrix} = \begin{pmatrix} -k\,i\,\alpha\,N_o & 0 & k\,E_o(1 - i\,\alpha) \\ 0 & -k\,i\,\alpha\,N_o & k\,E_o^*(1 - i\,\alpha) \\ -k\,E_o^*(N_o - 1) & -2\,k\,E_o\,(N_o - 1) & -(2\,k\,|A_o|^2 + \gamma) \end{pmatrix}\begin{pmatrix} e \\ e^* \\ n \end{pmatrix} \qquad (2.9)$$

{Footnote: e and e^* are not really independent but the current method for calculation will work}

The values in equations (2.2) and (2.3) are substituted into (2.9) in order to input selected values into the equations for simplification as well as selecting a range of values for the pumping (J) to observe how current affects the system. *Matlab* was used to solve for the above eigenvalues, which are independent of t. Graphical analysis was implemented in order to find the sign of the real part of the eigenvalues as a function of pumping vs. lambda (the eigenvalue solution). The following values were selected for the equation variables: $k = 1$, $\alpha = 1.01$, $\alpha_i = 1.02$, and $\gamma = 1.2$.

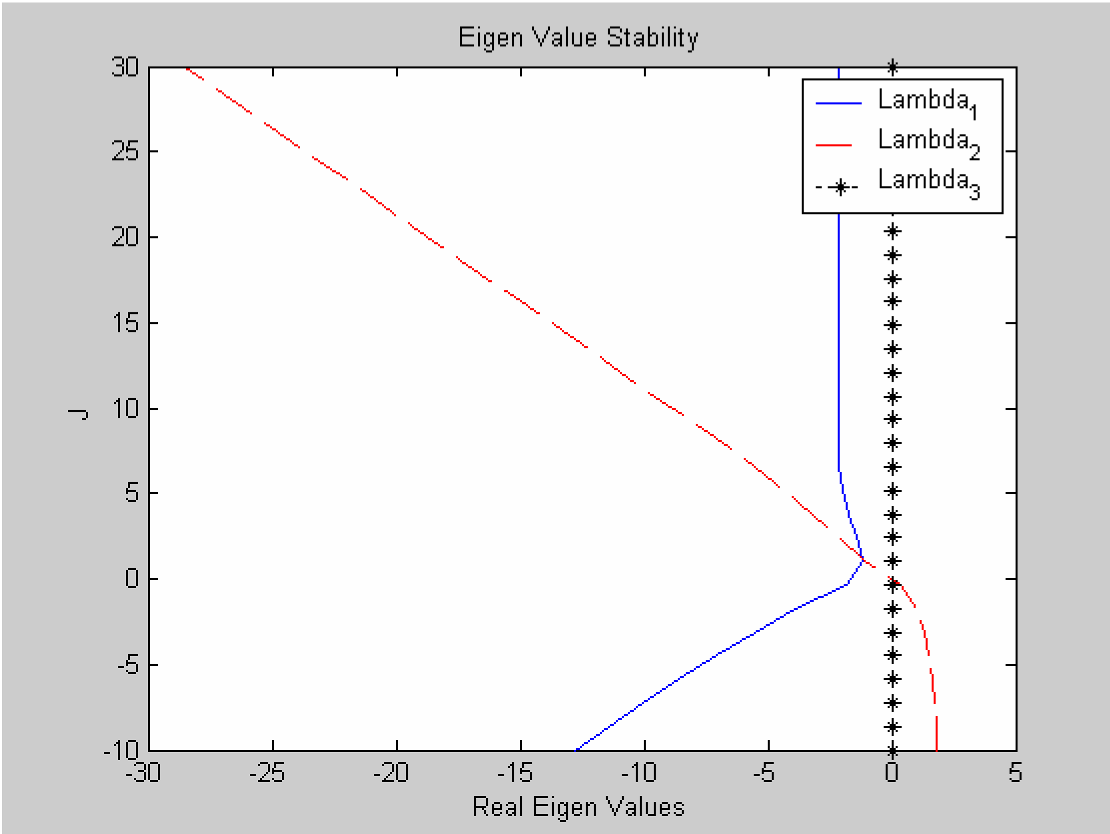


Figure 3.1 The stability analysis portraying eigenvalues' negative real parts.

Fig 3.1 is a graph that represents the real parts of the eigenvalues: lambda $_1$, lambda $_2$, and lambda $_3$ as a function of pumping. As seen in the graph, when the pumping is positive, corresponding to the laser being on, two of the eigenvalue solutions have a negative value and the other is exactly zero. The zero value solution for the eigenvalue is produced since it is purely imaginary, and the negative solutions show that the on fixed point (E_o , N_o)will in fact be stable. In brief, it has been established that the complex amplitude is stable when the intensity is stable for the laser in the on position.

In continuation, a feedback term is added to the Complex Amplitude System and analyzed with the above results.

Addition of Feedback Term and Analysis

Next, a feed back term is added to the original *Complex Amplitude System's* electric field equation in order to model the consequences of the mirror's feedback. This extra term possesses a reflectivity term (r), phase component (ϕ) and time delay (τ). This last term is used to model the light's time of travel between mirrors. The *Feedback System* is then defined as:

$$\frac{dE}{dt} = -\alpha_i\, E + k\,(N - 1 - i\,\alpha\, N)\,E + re^{i\theta}\,E[t - \tau\;] \qquad (1.9)$$

$$\frac{dN}{dt} = J - \gamma\, N - (N - 1)\,|E|^2 \qquad (1.10)$$

In the same form as in the previous analysis, this system is perturbed along with the known values of E_o and N_o. However, instead of breaking up E into its complex and real parts, it is simplified by separating its corresponding amplitude and phase components. Since all complex fields may be resolved into these two terms, this new avenue is also physically sound.

Specifically, let $E = A(t)\,e^{i\phi(t)} = Ae^{i\phi}$

Then (1.9) becomes:

$$\frac{d(Ae^{i\phi})}{dt} = e^{i\phi}\,\frac{dA}{dt} + iAe^{i\phi}\,\frac{d\phi}{dt} = -\alpha_i\,Ae^{i\phi} + k(N - 1 - i\alpha N)\,Ae^{i\phi} + re^{\theta}\,A[t-\tau]\,e^{i\phi[t-\tau]},$$

canceling common phase components and taking the feedback term as separable:

$$\Rightarrow \frac{dA}{dt} + iA\,\frac{d\phi}{dt} = -\alpha_i\,A + k(N - 1 - i\alpha N)\,A + re^{\theta}\,A[t-\tau]\,e^{i\phi[-\tau]},$$

separating real and imaginary parts provides separability between the amplitude and phase components:

$$\Rightarrow \frac{dA}{dt} = -\alpha_i\,A + k(N-1)\,A$$

$$\Rightarrow \frac{d\phi}{dt} = -k\alpha N + re^{\theta}\,A[-\tau]\,e^{i\phi[-\tau]},$$ the carrier density(N) equation remains:

$$\frac{dN}{dt} = J - \gamma N - 2\,k(N-1)\,A^2$$

These last three differential equations comprise the *Feedback System* which will also be perturbed in the quest for its own fixed points.

In analyzing this new system it will be beneficial to adapt a Laplace Transform for the differential equations due to the time delay. To begin the time delay is analyzed on its own:

$$
\begin{aligned}
&L\left[\frac{dy}{dt}\right] = L[y[t-1]], \\
&= sL[y] - y[0] = L[y[t-1]] \\
&\qquad = \int_0^{\infty} y[t-1]\, e^{-st}\, dt, \\
&\qquad = \int_0^{\infty} y[t-1]\, e^{-s(t-1)-s}\, dt, \text{ for } t-1 = u \\
&\qquad = e^{-s}\left(\int_{-1}^{\infty} y[u]\, e^{-su}\, du\right) \\
&\qquad = e^{-s} L[y[t]] + \int_{-1}^{0} y[t]\, e^{-st}\, dt \\
&\Rightarrow (s + e^{-s}) L[y] = y[0] - e^{-s} \int_{-1}^{0} y[t]\, e^{-st}\, dt \\
&\Rightarrow L[y] = \frac{y[0] - e^{-s} \int_{-1}^{0} y[t]\, e^{-st}\, dt}{s + e^{-s}}, \text{ for } y[t] = 0 \text{ for } t < 0 \\
&\Rightarrow L[y] = \frac{y[0]}{s + e^{-s}}, \text{ multiplying by } \frac{e^s}{e^s} \\
&\quad = \frac{e^s}{se^s + 1}\, y[0]
\end{aligned}
$$

Hence, by the Second Shift Theorem [3], we can find y[t]. Further, suppose that this function has the form $y = e^{st}$. Then:

$$
\begin{aligned}
\frac{dy}{dt} &= y[t-1] \\
se^{st} &= -e^{s(t-1)} \\
&= -e^{st} e^{-s} \\
\Rightarrow s = -e^{-s} &\Leftrightarrow se^s + 1 = 0
\end{aligned}
$$

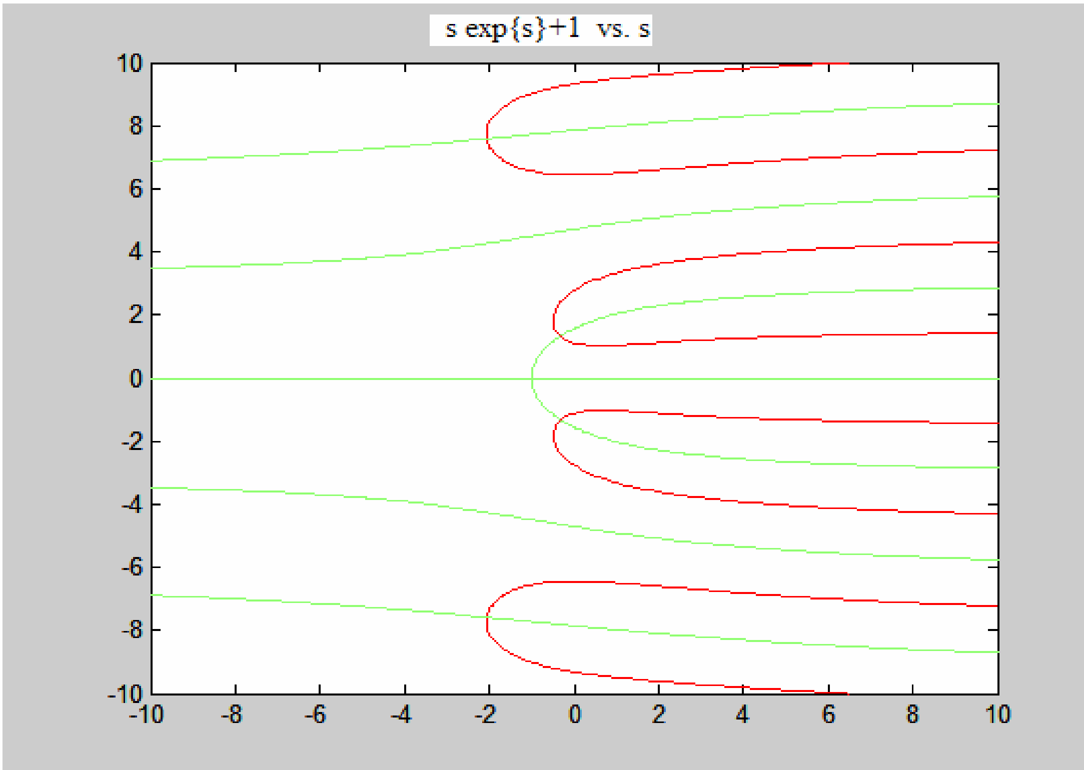
Hence, we must analyze the values of *s* which will make this analysis valid. Since a closed form solution of the last statement does not exist, a graphical analysis is performed on the complex variable *s*. To do so, first consider:

$$
\begin{aligned}
&s = \sigma + it, \ni \sigma, t \in \mathbb{R} \\
&\Rightarrow se^s + 1 = (\sigma + i\tau)\, e^{\sigma + i\tau}, \text{ by Euler's relations:} \\
&\qquad = (\sigma + i\tau)\, e^{\sigma} (\cos[\tau] + i\sin[\tau]) + 1 \\
&\qquad = e^{\sigma}(\sigma\cos[\tau] - \tau\sin[\tau]) + ie^{\sigma}(\tau\cos[\tau] + \sigma\sin[\tau])
\end{aligned}
$$

To obtain an improved description of what s may be the last expression is decoupled into its real and imaginary parts:

$$
\begin{aligned}
\mathrm{Re}\{se^s + 1\} &\Rightarrow e^{\sigma}(\sigma\cos[\tau] - \tau\sin[\tau]) = -1 \\
\mathrm{Im}\{se^s + 1\} &\Rightarrow \tau\cos[\tau] + \sigma\sin[\tau] = 0
\end{aligned}
$$

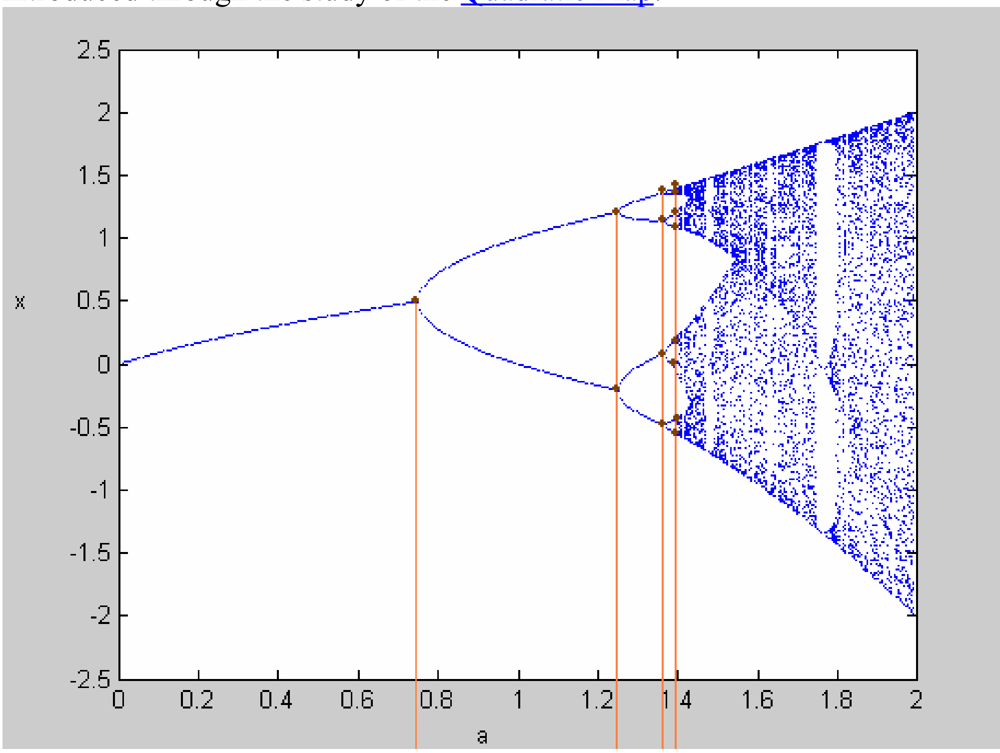
Since the above system is not linear and possesses no closed form solution, a graphical analysis is performed by simultaneously plotting these two equations. The intersections of the two functions will indicate acceptable values of s for the Laplacian analysis of the problem at hand.



From this graph it is seen that the intersection of the green and red lines indicate a periodic solution for values *s*. To establish that the only intersections on this graph exist on the negative cone, a complex variables analysis is performed. Exploiting the fact that these equations describing s are holomorphic a contour integral is taken around a simply connected region, which does not include (cuts) these solutions. Hence, if this calculation yields zero, then it will in fact be the case that the conditions on *s* lie within a bounded region.

The continuation of this analysis will be explored in the future.

At this point the beginnings of chaos in what is to be the complete feedback system is introduced through the study of the Quadratic Map:



Future Plans

Kenny Headington

The first step in further developing this research is to analytically solve for simple solutions of the individual chaotic system with feedback. The analytical solutions will be the basis for numerical solutions concerning the stability of the system. Then I will take two copies of the original system and couple them. Through coupling the chaotic laser systems the synchronization of the coupled system may then be analyzed.

This research has further developed my mathematical skill and has intrigued a further interest into the mathematics field. Because of this research I now plan to add more math classes to my curriculum and possibly work towards a math major utilizing the techniques acquired from this research project. I am looking forward to forming new data on the previously derived concept of coupled lasers with chaotic nature. I would also like to thank Soneson, professor Indik, and the University of Arizona mathematics department for giving me the opportunity to develop this research.

Ivan Barrientos

In the continuation of this project, I would seek to complete the complex variable analysis under the Laplacian analysis section. In addition, I would like to accurately model the dynamics of a single laser being self-coupled by a single plane mirror. After establishing the necessary length of the cavity for stability in resonance. Thereafter, the dynamical system of two interacting lasers could be studied. Finally, a conjecture may be established for the physical requirements necessary to couple a realistic amount of lasers. In the end, the precision of these conclusions may possibly be quantified by quasi-likelihood statistics.

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