GELFAND'S THEOREM

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This paper is a proof of the first part of Gelfand's Theorem, which states the following: Every compact, Hausdorff space, is equivalent to the spectrum of the C*-algebra of continuous, complex valued functions on that space. The proof utilizes ideas from abstract and linear algebra, topology, and analysis. This project was supported by the mathematics department at the University of Arizona, and an NSF VIGRE grant.

Part I of Gelfand's Theorem is the following statement:

$$X \cong SpecC(X)$$

Where X is a compact, Hausdorff space and $C(X)$ denotes the C*-algebra of continuous, complex functions on the set, X. We proceed, carefully, by showing that there exists a bijective function from X onto Spec $C(X)$. Then, we show that this relation is actually a homeomorphism between compact, Hausdorff spaces.

First, we provide some background.

We begin by showing that $C(X)$ is a complex vector space under pointwise addition, and a complex algebra under pointwise multiplication.

$C(X)$ is a vector space over the field $\mathbb{C}$.
Proof. We verify the axioms for a vectorspace.

V1. *Commutative under addition*
We consider $f, g \in C(X)$. Then, $\forall x \in X, (f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$, since + is defined using pointwise addition.

V2. *Associative under Addition*
Using pointwise addition,
$\forall f, g, h \in C(X), \forall x \in X((f+g)+h)(x) = (f+(g+h))(x)$.

V3. *There is an additive identity*
$\forall x \in X$, Let $g(x) = 0$. Then g is the additive identity.

V4. *Additive Inverses Exist*
$\forall f \in C(X), \forall x \in X, \exists g \in C(X) \ni g(x) = -f(x)$.

V5. *There is a scalar with multiplicative identity*
$\forall f \in C(X), \forall x \in X, (1 \cdot f)(x) = f(x)$.

V6. *Associative under Multiplication*
$\forall f \in C(X), \forall x \in X, (ab)(f)(x) = a(b \cdot f)(x)$

V7. *Distributive over vectors*
$\forall f, g \in C(X), \forall x \in X, (a)(f+g)(x) = a(f)(x) + (g)(x)$

V8. *Distributive over scalars*
$\forall f \in C(X), \forall x \in X, (a+b)(f)(x) = a(f)(x) + b(f)(x)$

Thus, $C(X)$ is a vector space. Now, we are interested in working with $C(X)$ as a $\mathbb{C}$-algebra. We know that an algebra is a vector space equipped with a multiplication that is associative for scalars and vectors, and commutes with scalars. Thus, we define

$$(f \cdot g)(x) = f(x) \cdot g(x)$$
and all of the conditions are clearly satisfied.

We go further, and verify that $C(X)$ is actually a C^*-algebra, by showing that $\forall f, g \in C(X), \forall \mathrm{a} \in \mathbb{C}$,

$$(f^*)^* = f, \quad (f+g)^* = f^* + g^*, \quad (fg)^* = g^* f^*, \quad (\mathrm{a}f)^* = \overline{\mathrm{a}} f^*$$

These identities are obvious when we recognize that each element of $\mathbb{C}(X)$ can be decomposed: $f(x) = f_1(x) + i \cdot f_2(x)$. Then, the *-operation is simply complex conjugation.

Finally, we verify that $C(X)$ is a unital C^*-algebra, by showing that there exists an element, $f \in C(X) \ni \forall g \in C(X), \forall x \in X, (f \cdot g)(x) = g(x)$. This is the element $f \ni$

$$\forall x \in X, f(x) = 1.$$

Now that we have $C(X)$ as a unital C^*-algebra, we would like to define a norm on $C(X)$, such that $C(X)$ is complete with respect to the metric induced from the norm.

Define $\| \; \| : C(X) \to \mathbb{R}$ such that

$$\|f\| = \max_{x \in X} |f(x)|$$

Clearly, this is a norm, provided that the max exists. However, since X is and also satisfies $\forall f, g \in C(X), \|f \cdot g\| \leq \|f\| \cdot \|g\|$. $C(X)$ is complete with respect to this metric, though the proof is omitted.

Now, we're ready to consider the relation between the set X and the Spec$C(X)$. Spec$C(X)$ is defined as the set of all non-zero C^*-homomorphisms that map elements of C(X) to $\mathbb{C}$. We will show that there is an injective correspondence between points of these sets.

To prove that there is an injection, we will actually prove that every non-zero C^*-homomorphism, called ϕ, returns the value $f(x)$ for some point $x \in X$. Then, the injective relation between X and set of all ϕ's is obvious: for every x, we have $\phi(f) = f(x)$, which uniquely determines ϕ.

If $\phi : C(X) \to \mathbb{C}$ is a non-zero C^*-homomorphism, then $\exists x \in X \ni \forall f \in C(X)$,

$$\phi_x(f) = f(x)$$

Proof. We proceed by contradiction. Assume, then, $\forall x \in X, \exists f \in C(X) \ni \phi(f) \neq f(x)$. Since $\phi : C(X) \to \mathbb{C}$ sends an infinite dimensional space to a one dimensional space, we know that the kernel of ϕ, the set $\{ f \in C(X) \,|\, \phi(f) = 0 \}$, is non-empty. Consider an $x \in X$. By our assumption, $\exists f_x \ni \phi(f_x) = 0$ and $f_x \neq 0$. Then $f_x \in \ker(\phi)$. We know the kernel of a homomorphism is an ideal (Appendix 1), so f_x is in the ideal, $\ker(\phi)$, of

$C(X)$. Now, consider $\overline{f}_x$. Since ϕ is a C*-homomorphism, it is clear that $\phi(\overline{f}_x) = \phi(f_x)^* = 0^* = 0$, so $\overline{f}_x$ is also in the ideal. Now, since $f_x(x) \neq 0$, it must be that $\overline{f}_x(x) \neq 0$. Now, we consider the product, $f_x \cdot \overline{f}_x = \left| f_x \right|^2 \in \ker(\phi)$, since $\phi(\left| f_x \right|^2) = \phi(f_x \cdot \overline{f}_x) = \phi(f_x) \cdot \phi(\overline{f}_x) = 0$. Now, we can use the compactness of X to show that $1 \in \ker(\phi)$.

For each $x \in X$, we can find a continuous function $\left| f_x \right|^2$, that is real and non-zero for some neighborhood about x. The union of all such neighborhoods is an open cover of the space, X. Then, since X is compact, we know that there exists a finite subcover which also covers X. We consider this finite subcover. So we have a set of neighborhoods about this finite set of functions, $\left\{ f_{x_1}, ..., f_{x_n}, \right\}$. Let $f = \left| f_{x_1} \right|^2 + ... + \left| f_{x_n} \right|^2$.

Since ϕ is linear, it is must be that $\phi(f) = \phi(\left| f_{x_1} \right|^2) + ... + \phi(\left| f_{x_n} \right|^2) = 0$, so $f \in \ker(\phi)$.

Now define a function, $f^{-1} = \left(\frac{1}{f(x)} \right) \forall x \in X$. Since f is continuous, and positive, so is f^{-1}. Now, $\ker(\phi)$ contains the multiplicative identity, since $\phi(1) = \phi(f^{-1} \cdot f) = \phi(f^{-1}) \cdot \phi(f) = 0$. Thus, $\ker(\phi) = C(X)$ (Appendix 2). However, this means that $\phi : C(X) \to \mathbb{C}$ is a zero homomorphism. This contradicts our hypothesis that ϕ is non-zero. Thus, we conclude that every non-zero C^*-homomorphism must be evaluation at a point for some point $x \in X$.

At this stage in the proof, we have demonstrated a one to one correspondence between the compact, Hausdorff space, X, and the set of all non-zero C*-homomorphisms, $\operatorname{Spec} C(X)$. Now, we would like to demonstrate that there is a homeomorphism between these compact, Hausdorff spaces. Of course, this requires that we impose a Hausdorff topology on $\operatorname{Spec} C(X)$.

We notice that $\operatorname{Spec} C(X)$ is actually a subset of $C(X)^\vee$, the dual space of $C(X)$. There is a natural topology to put on a dual space, called the weak-* topology. This topology has the following bases:

$$N(\phi_0 : S, \varepsilon) := \left\{ \phi \in C(X)^\vee : \left| \phi(f) - \phi_0(f) \right| < \varepsilon \ \forall f \in S \underset{finite}{\subseteq} C(X) \right\}$$

Thus, each basis depends on three parameters: ϕ_0, ε, S. We would like to show that this weak-* topology on $C(X)^\vee$ is Hausdorff.

Proof. To show this topology is Hausdorff, we consider $\phi_1, \phi_2 \in C(X)^\vee \ni \phi_1 \neq \phi_2$. We want to show that there are two neighborhoods containing ϕ_1, ϕ_2 respectively that have null intersection. We will do this constructively, by showing that there are neighborhoods which meet these requirements.

First, we define the parameters. Let $N_1 = N(\phi_1 : \{g\}, \varepsilon)$, and let $N_2 = N(\phi_2 : \{g\}, \varepsilon)$ for any g such that $\phi_1(g) \neq \phi_2(g)$. Then, let $\varepsilon = \frac{|\phi_1(g) - \phi_2(g)|}{2}$.

Thus, we have the following neighborhood:

$$N_1 = \left\{ \phi \in C(X)^\vee : |\phi(g) - \phi_1(g)| < \frac{|\phi_1(g) - \phi_2(g)|}{2} \right\}$$

$$N_2 = \left\{ \phi \in C(X)^\vee : |\phi(g) - \phi_2(g)| < \frac{|\phi_1(g) - \phi_2(g)|}{2} \right\}$$

By our choice of g, it is clear that both neighborhoods are non-empty, for they must contain ϕ_1, ϕ_2 respectively, and they have a null intersection.

Now, we would like to define a norm on $C(X)^\vee$. Consider any $\phi \in C(X)$. It can be verified that there is a bound on $\frac{|\phi(f)|}{\|f\|}$ that is independent of $f \in C(X)$. Then, we define the norm $\| \ \| : C(X)^\vee \to \mathbb{R}$ as

$$\|\phi\| = \sup_{f \neq 0} \frac{|\phi(f)|}{\|f\|}.$$

This is a norm on $C(X)^\vee$.

Proof. We verify that this definition satisfies the conditions. $\frac{|\phi(f)|}{\|f\|} > 0 \ \forall f \in C(X)$ by definition of absolute value and the norm of f. It follows that the supremum must also be positive. Now, consider $\alpha \in \mathbb{C}$. Then clearly $\| \alpha \cdot \phi \| = |\alpha| \cdot \|\phi\|$ by properties of suprema. Finally, since suprema respect the triangle inequality, it is clear that $\forall \phi_1, \phi_2 \in \mathbb{C}(X)$,

$$\|\phi_1 + \phi_2\| \leq \|\phi_1\| + \|\phi_2\|$$

In order to prove that $\operatorname{Spec} C(X)$ is a *compact* Hausdorff space, we will need to show that it is a closed subset of the unit ball, which is compact in the weak-* topology by the Banach-Alaoglu Theorem.

$\operatorname{Spec} \mathbb{C}(X)$ is completely contained inside the unit ball in $C(X)^\vee$.

Proof. We consider an arbitrary $\phi_x \in \text{SpecC(X)}$. Then ϕ is given by evaluation at some $x \in X$. We want to show that the norm of ϕ is ≤ 1. Specifically,

$\|\phi_x\| = \sup_{f \neq 0} \frac{|\phi_x(f)|}{\|f\|}$, where we defined $\|f\| = \max_{x \in X} |f(x)|$. Combining these expressions, we have $\|\phi_x\| = \sup_{f \neq 0} \frac{|\phi_x(f)|}{\max_{x \in X} |f(x)|}$. Clearly, $\|\phi_x\| \leq 1$ iff $\frac{|\phi_x(f)|}{\max_{x \in X} |f(x)|} \leq 1 \ \forall f \in \mathbb{C}(X)$. But this result is obvious, because $|\phi_x(f)| = |f(x)| \leq \max_{x \in X} |f(x)|$ by definition of maximum.

Spec $C(X)$ is closed.
Proof.

To show that Spec $C(X)$ is closed, we show that its complement is open in the weak-* topology. Specifically, if $\phi \in C(X)^\vee$ and $\phi \notin$ Spec $C(X)$, then it must be that $\phi = 0$ or that the very strict, multiplicative condition is not satisfied, that is $\phi(f) \cdot \phi(g) \neq \phi(f \cdot g)$, for $f, g \in C(X)$. So, to prove that the complement of Spec $C(X)$ is open, we must find a neighborhood, $N(\phi_1 : \{f, g, fg\}, \varepsilon > 0)$, which has null intersection with Spec $C(X)$. This can be verified if we take $N(\phi : \{f, g, fg\}, \phi(f \cdot g) - \phi(f) \cdot \phi(g))$. (This choice of epsilon seems to be the best choice; however, I have not formalized my argument here)

Because Spec $C(X)$ is contained in the unit ball, and is closed, Spec $C(X)$ is compact by the Banach-Alaoglu theorem.

Thus, we have established that Spec $C(X)$ is a compact, Hausdorff space. Now, we want to find a homeomorphism between X and Spec $C(X)$ to complete this part of Gelfand's Theorem.

Define $\Phi :$ Spec $C(X) \to X$ by $\Phi(\phi_x) = x$. We will show that a continuous bijection between compact Hausdorff spaces is a homemorphism. So, first we need to verify that Φ is continuous.

Proof.

To show that Φ is continuous, we will show that the pre-image of any closed set in X is closed in Spec $C(X)$. So, we consider a closed set, C in X. We will show that $\Phi^{-1}(C)$ is closed by showing that its complement, intersect Spec $C(X)$ is open. So, specifically we want $\{\phi_x \in \text{Spec}C(X) \mid \Phi(\phi_x) \notin C\}$ to be an open set. This set corresponds to the set $\{y \in X \mid y \notin C\}$. From Uryshon's lemma, define a function $f_y(y) = 0$ and $f_y(z) = 1 \forall z \in C$. So we consider a neighborhood,

$$N(\phi_y : \{f_y\}, \frac{1}{2}) = \left\{ w \in \text{Spec}C(X) \mid \left| w(f_y) - \phi_y(f_y) \right| < \frac{1}{2} \right\}$$

Then, using our definition of f_y, this is the neighborhood

$$N(\phi_y : \{f_y\}, \frac{1}{2}) = \left\{ w \in \text{Spec}C(X) \mid |w(f_y)| < \frac{1}{2} \right\}$$

So we have showed that the complement of $\Phi^{-1}(C)$ is closed. It follows that Φ is continuous.

Φ is a homeomorphism between compact Hausdorff spaces.

Proof. To prove this result, we must show that Φ is bijective, continuous, and that Φ^{-1} is continuous. Since we have already shown the first two conditions are satisfied, it remains to show that Φ^{-1} is continuous.

Consider a closed subset A of Spec $C(X)$. Then A is compact since it is a closed subset of a compact space. Then, $\Phi(A)$ is compact because the image of a compact space under a continuous map is compact. Finally, $\Phi(A)$ is closed in X since every compact subspace of a Hausdorff space is closed.

So we have established that every closed subset in the domain space is closed in the image space. This verifies that Φ^{-1} is continuous. It immediately follows that Φ is a homeomorphism between compact Hausdorff Spaces.

This proves that $X \cong \text{SpecC}(X)$, and completes Part I of Gelfand's Theorem.

PART II.

We now want to prove an analogous relationship for abelian C*-algebras. Let A be an abelian C* algebra. Then we will show

$$A \cong C(\text{ Spec } A)$$

To show that A is an isometric *-isomorphism (of C*-algebras) we must show that there is some map $f : C(\text{ Spec } A) \to A$ which is

1) Isometric. This means that it preserves distances.
$$\forall x_i \in X, \| f(x_1) - f(x_2) \| = \| x_1 - x_2 \|$$
2) A *-Isomorphism.
   a) The star means that f preserves conjugation
   $$(\forall x \in X, f(x^*) = f(x)^*)$$
   b) An Isomorphism is a bijective homomorphism, thus
      i) f is injective and surjective
      ii) f preserves the structure of the algebra
      $$\forall x_i \in X, f(ax_1 + x_2) = af(x_1) + f(x_2)$$
      $$f(x_1 \cdot x_2) = f(x_1) \cdot f(x_2)$$

This is the general strategy to be employed in Part II. In particular, we will show that f satisfies the above conditions if we consider f to be the Gelfand transformation

$$\hat{x} : SpecC(X) \to \mathbb{C} \quad \hat{x}(l) := l(x) \quad l \in Spec(C(X))$$

Before we continue with this program, it is essential that we mention an important theorem, which we will use later on. It is the analogue to the Spectral Theorem in linear algebra, which says that if some linear operator is self-adjoint, then its eigenvalues are real. In this case, it can be shown (though the proof is omitted) that if $f = f^*$ for some $f \in C(X)$, then $\sigma(f) := \left\{ \lambda \in \mathbb{C} \mid \lambda \cdot 1_{C(X)} - f \notin C(X)^{\times} \right\} \in \mathbb{R}$. Here, $\sigma(f)$ is Spectrum of f.

Proposal for Summer 2004:

I propose to finish the program of the proof of Gelfand's Theorem. Then, with the proof completed, I would like to analyze the space of all Penrose tilings from the standard topological point of view, and secondly from a "non-commutative" point of view as discussed in Tamas Tasnadi's paper, *Penrose Tilings, Chaotic Dnyamical Systems and Algebraic K-Theory*. We will discover the necessity of studying such systems from this point of view, as it has been shown that studying Penrose tilings from a topological point of view is problematic.

To provide a basic idea of how we study Penrose Tilings, we remember that every Penrose Tiling can be generated by successive double decompositions, followed by inflation. This yields the following transformation

$$L \rightarrow 2L + S$$
$$S \rightarrow S + L$$

Where L,S are long and short prototiles, as discussed in my previous paper. Then, each Penrose Tiling can be completely characterized by a binary sequence of L's and S's, and the following grammar rule,

$$X_i = S \Rightarrow X_{i+1} = L$$

We will be able to show, using the above considerations, a metric on the set of all Penrose Tilings. This will allow us to compare two "similar" tilings and find out how closely related they are. We will also be able to show that the set of all Penrose tilings is an uncountable set.

APPENDIX

1) *The kernel of a homomorphism is an ideal*.

Proof. Suppose we are given a homomorphism, $\phi : A \rightarrow B$, where A,B are algebras. Then, $\ker(\phi)$ is an ideal of A.

Consider $a_1, a_2 \in \ker(\phi)$. By definition of the kernel, $\phi(a_1) = \phi(a_2) = 0_B$. Then $\phi(a_1 - a_2) = \phi(a_1) \ominus \phi(a_2) = 0_B$, since homomorphisms respect subtraction. Thus, $a_1 - a_2 \in \ker(\phi)$.

Now consider, $a_1 \in \ker(\phi), a \in A$. Then, $\phi(a_1 \cdot a) = \phi(a_1) \odot \phi(a) = 0_B \odot \phi(a) = 0_B$. Thus, $a_1 \cdot a \in \ker(\phi)$.

This completes the proof that $\ker(\phi)$ is an ideal of A.

2) *If an ideal contains a multiplicative inverse, it must be the whole algebra*.

Proof. Consider an algebra, A, with an ideal K and multiplicative identity 1_K. A=K.

Consider an $a^{-1} \in A$. Since K is ideal, $a^{-1} \cdot a \in K \Rightarrow 1 \in K$. Thus, $\forall b \in A$, $b \cdot 1 \in K$ and $1 \cdot b \in K$. Thus, $A \subseteq K$. Clearly, $K \subseteq A$. It follows that A=K.