

Undergraduate Research Project

Department of Mathematics
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Dynamic Systems and Chaos

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This work is a continuation of the work done in spring 2004 with Ivan Barrientos and Dorian Smith. Significant parts of this report are taken from that research.

Chaos Theory

Chaos is a fundamental property that possesses nonlinearity and a sensitive dependence on initial conditions. Because of the nonlinearity in a chaotic system it becomes very difficult to make accurate predictions about the system over a given time interval. Weather forecasting is an example of how chaos theory effects the accuracy of predictions over a given time interval. Through analyzing a weather pattern over time, meteorologists have been able to make better predictions of future weather based on this

theory. Another area in the domain of chaos is the behavior of lasers. Specifically, the emission of a laser is affected by chaos due to feedback in the system. Feedback originates from the reflections in the optical cavity of the laser, and is amplified through multiple reflections and emissions. This feedback becomes chaotic as it leaves the optical cavity and enters the external cavity of the laser where a time delay takes place. Hence, the intensity of the emitted beam may be modeled by the chaotic properties of the external feedback. Therefore, in order to make predictions about a laser's intensity over an amount of given time, chaos theory should be studied and applied to the laser rate equations. [1]

Stability is an important concept when studying systems over a given time interval. A system is considered stable when a condition converges toward a single point within a set range. On the other hand, a system becomes unstable when conditions diverge from a fixed point and depart from this range. Further, when the system diverges and splits, a more complicated system is created. The locations of these splits are called bifurcation points. As the system progresses with time it exponentially develops more bifurcation points. These special points can be related to the chaotic behavior of two synchronized laser systems.

Synchronization

The main goal of the current research on the chaotic properties of lasers is to establish the necessary conditions for synchronizing coupled lasers. The coupling of the lasers combines two individually chaotic systems into one system with a superposition of emissions. In order to analyze a chaotic coupled laser system, quadratic maps with iterations will be created and analyzed using semiconductor rate equations, which is illustrated in Fig 1.1 below.

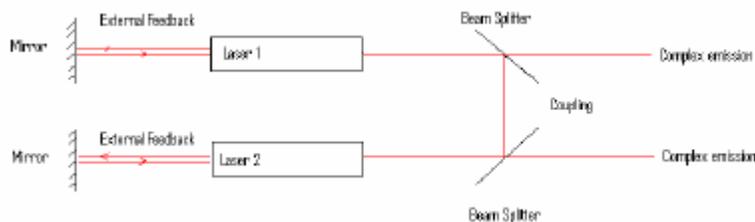


Figure 1.1 A model of two lasers being coupled by reflecting mirrors and beam

The Semiconductor Laser

Lasers have redefined the scientific world, commerce, and everyday living over the last century. Lasers may be found anywhere from scientific laboratories to supermarkets; and therefore, should be further explored and researched in order to progress as a technological society. Solid state, semiconductor, and gas lasers are just a few of the many different types of lasers available on the market today. Each of these different types of lasers is important for different applications based upon the desired result and cost. This research project examines the properties of semiconductor lasers because of their versatility and cost-efficiency.

First, it is important to understand what a laser is and why it is a desirable light emitting source. Laser is an acronym for Light Amplification by Stimulated Emission of Radiation. As such, a laser amplifies light in order to create a desired emission. Light is amplified through the properties of p-n junctions, electron-hole pairs, optical gain, and population inversion. Here, the chaotic characteristics of laser emission will be explored. Understanding the chaotic nature of the laser will lead to accurate predictions of the emitted irradiance of the laser, which is necessary when studying more complex systems with feedback and coupling.

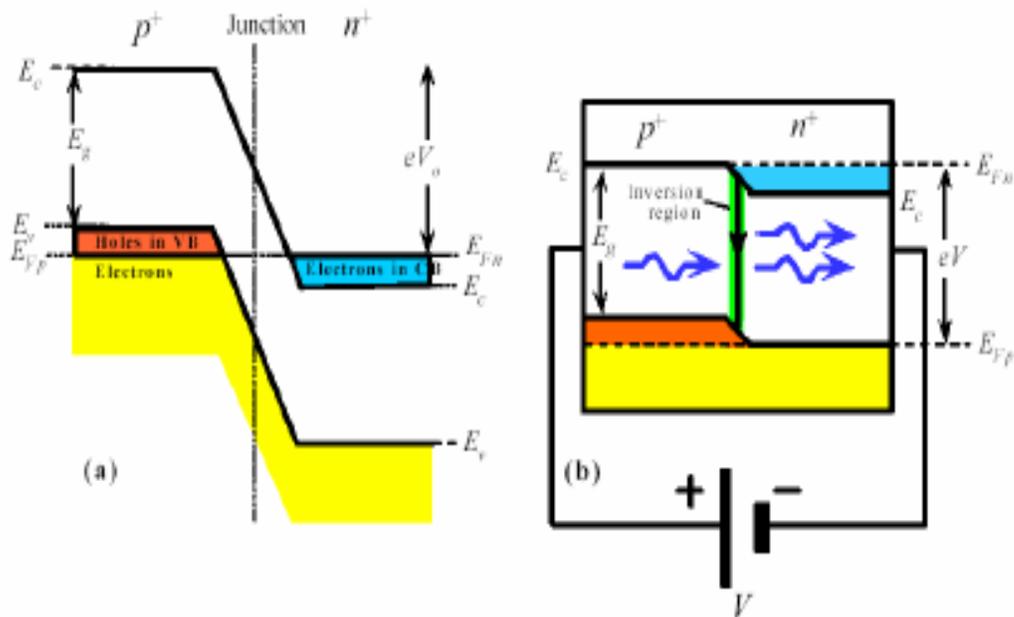
How a Laser Emits Light

A semiconductor laser is made up of several different components which lead to the production of optical gain; whereby yielding stimulated emission. The basic components of the laser consist of a current source, a semi-conductive material, and an optical cavity. Each of these components controls the stimulated emission of the laser and need to be understood in order to perform well-founded computational analysis.

Semiconducting Material

A material is considered semiconducting if it has conductance, but its conductance must be less than that of a metal. The semiconducting material used in a semiconductor laser is composed of electron-hole pairs defining the n-type and p-type junctions of the material. For a material to be either n- or p-type an impurity must be inserted into the crystal lattice. This process is termed ‘doping’. If a larger amount of electrons are inserted in the lattice of the material, it becomes n-type. In contrast, if the doping introduces a larger amount of holes, then the lattice is considered to be p-type. These properties allow for stimulated emission to occur through using a current source to create pumping.

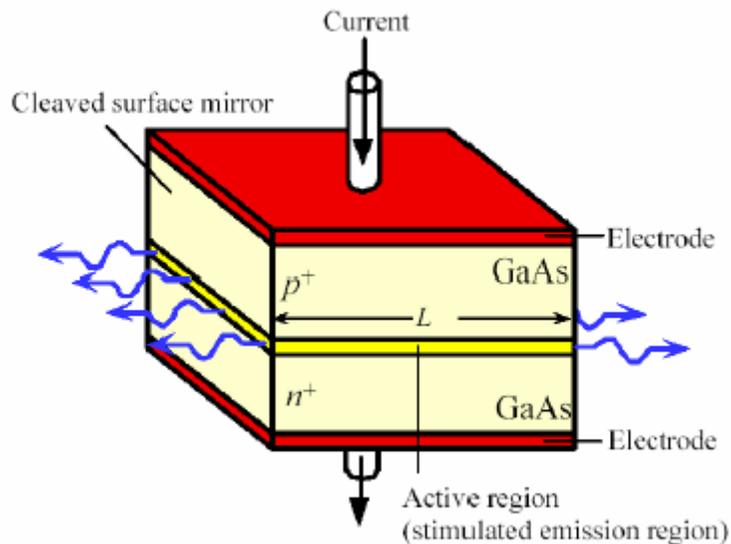
[2][3]



Population Inversion

In order for the laser to emit light population inversion must be present in the system. Population inversion means that there are more electrons available to make a “down” transition and amplify the light than are available to make an up transition and absorb light. This process occurs as a consequence of pumping in the optical cavity. [2]

An optical cavity is necessary to produce a stimulated emission through pumping. This cavity is a region composed of two approximately parallel mirrors separated by a defined distance. The semiconducting material used in the laser forms the optical cavity in the case of the semiconductor laser. Moreover, the parallel edges of the material have mirror like properties themselves and are separated by the length of the material in order to form the optical cavity, which then produces a positive feedback. [2]



Gain is a measurement that is determined by the length of the optical cavity and the number of reflected passes through the semiconducting material. For each pass through the optical cavity a loss occurs due to the mirrors that are proportional to the gain. When pumping is applied to the semiconducting material, the gain increases for each pass through the optical cavity. Population inversion occurs when the gain reaches a value higher than the loss from reflection. [2]

Finding Solutions for the Laser on Case

(The proceeding introduction was largely taken from spring 2004 and the following information is a new continuation of that work.)

The first step for analyzing the stability of the electric field envelope and the carrier density of the semiconductor laser is to find solutions for the laser system for the on case. The primary equations describing the electric field envelope and the carrier density are:

$$\frac{d}{dt}E = -\alpha_{int} \cdot E + g \cdot [(N - 1) - i \cdot \alpha \cdot N] \cdot E \quad [1.1]$$

$$\frac{d}{dt}N = J - \gamma \cdot N - 2 \cdot g \cdot (N - 1) \cdot |E|^2 \quad [1.2]$$

From these primary equations we will look for a solution for $E_o(t)$ where $E_o(t) = \sqrt{I_o} \cdot e^{i \cdot \theta \cdot t}$ and solve for the variables I_o , θ , and N_o .

$$\frac{d}{dt}E_o = i \cdot \theta \cdot \sqrt{I_o} \cdot e^{i \cdot \theta \cdot t} \quad [1.3]$$

Through the substitution of $E_o(t)$ into [1.1], the following equation is formed.

$$\frac{d}{dt}E_o = -\alpha_{int} \cdot \sqrt{I_o} \cdot e^{i \cdot \theta \cdot t} + g[(N_o - 1) - i \cdot \alpha \cdot N_o] \cdot \sqrt{I_o} \cdot e^{i \cdot \theta \cdot t} \quad [1.4]$$

Equations [1.3] and [1.4] will now be set equal to each other to find solutions to the variables θ and N_o using real and imaginary relationships.

$$i \cdot \theta = -\alpha_{int} + g \cdot [(N_o - 1) - i \cdot \alpha \cdot N_o] \quad [1.5]$$

The variable θ may now be solved for by analyzing the imaginary part of [1.5].

$$i \cdot \theta = -i \cdot g \cdot \alpha \cdot N_o, \text{ Therefore}$$

$$\theta = -g \cdot \alpha \cdot N_o \quad [1.6]$$

The next step is to solve for N_0 by analyzing the real part of [1.5].

$$0 = -\alpha_{int} + g \cdot (N_0 - 1), \text{ Therefore}$$

$$N_0 = 1 + \frac{\alpha_{int}}{g} \quad [1.7]$$

Because $|E|^2$ is equal to I_0 and the solution for N_0 has been found, equation [1.2] will be used to find the solution for I_0 .

$$\frac{d}{dt} N_0 = 0 = J - \gamma \cdot N_0 - 2g \cdot (N_0 - 1) \cdot I_0$$

$$I_0 = \frac{J - \gamma \cdot \left(1 + \frac{\alpha_{int}}{g}\right)}{2 \cdot \alpha_{int}}$$

Therefore, [1.8]

$E_0(t)$ and N_0 may now be written as solutions to the semiconductor laser for the on case:

$$E_0(t) = \sqrt{I_0} \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t} \quad [1.9]$$

$$N_0 = 1 + \frac{\alpha_{int}}{g}$$

Since $I_0 \geq 0$

$$J_{\text{critical}} = \gamma \cdot \left(1 + \frac{\alpha_{int}}{g}\right)$$

When $J > J_{\text{critical}} = 0.002$ the laser will switch on.

Stability for the Lasing Solutions with the Original Variables

The following calculations are based on the perturbation analysis done in [Spring 2004](#) to determine the stability of the electric field envelope and the carrier density for the on solution, but the E_0 formula has been changed to [1.9]. We will now perturb the solutions:

$$E = E_0(t) + e \quad [1.10]$$

$$N = N_0(t) + n \quad [1.11]$$

Where e and n are very small.

We will now rewrite equations [1.1] and [1.2] with the perturbation from [1.10] and [1.11].

$$\frac{d}{dt}E = \frac{d}{dt}E_0 + \frac{d}{dt}e = -\alpha_{int} \cdot (E_0 + e) + g[(N_0 + n - 1) - i \cdot \alpha \cdot (N_0 + n)] \cdot (E_0 + e) \quad [1.12]$$

$$\frac{d}{dt}N = \frac{d}{dt}N_0 + \frac{d}{dt}n = J - \gamma \cdot (N_0 + n) - 2 \cdot g \cdot (N_0 + n - 1) \cdot (|E_0 + e|)^2 \quad [1.13]$$

Where ee^* is dropped after linearization, $|E_0 + e|^2 \approx |E_0|^2 + 2 \operatorname{Re}\{E_0 e^*\}$

The next step is to simplify and group terms for equation [1.12], which is done through using equation [1.1].

$$\frac{d}{dt}E_0 + \frac{d}{dt}e = -\alpha_{int} \cdot E_0 - \alpha_{int} \cdot e + g[(N_0 - 1) - i \cdot \alpha \cdot N_0] \cdot E_0 + g[(N_0 - 1) - i \cdot \alpha \cdot N_0] \cdot e + g \cdot n \cdot (1 - i \cdot \alpha) \cdot E_0 + g \cdot n \cdot (1 - i \cdot \alpha) \cdot e$$

This is then reduced to the following equation using equation [1.1] and dropping the second order term.

$$\frac{d}{dt}e = [g \cdot [(N_0 - 1) - i \cdot \alpha \cdot N_0] - \alpha_{int}] \cdot e + g \cdot E_0 \cdot (1 - i \cdot \alpha) \cdot n \quad [1.14]$$

The complex conjugate to [1.14] is the following equation.

$$\frac{d}{dt}e^* = [g \cdot [(N_0 - 1) + i \cdot \alpha \cdot N_0] - \alpha_{int}] \cdot e^* + g \cdot E_0 \cdot (1 + i \cdot \alpha) \cdot n \quad [1.15]$$

We will now find the solution to the perturbation of the carrier density equation [1.2] using the same process as above. The grouped perturbed equation is:

$$\frac{d}{dt} N_0 + \frac{d}{dt} n = J - \gamma \cdot N_0 - 2 \cdot g \cdot (N_0 - 1) \cdot \{I_0 + e \cdot E_0^* + E_0 e^*\} - \gamma \cdot n - 2g \cdot n \cdot \{I_0 + e \cdot E_0^* + E_0 e^*\}$$

This is then reduced to the following equation by using equation [1.2] and dropping higher order terms.

$$\frac{d}{dt} n = [-2 \cdot g \cdot E_0^* \cdot (N_0 - 1)] \cdot e + [-2 \cdot g \cdot E_0 \cdot (N_0 - 1)] \cdot e^* + [-(2gI_0 + \gamma)] \cdot n \quad [1.15]$$

The usual method to find the stability of the $E_0(t)$, N_0 solution is to find eigenvalues. As we will later see, this is not quite right in this situation. In order to accomplish this equations [1.14], [1.15], and [1.16] are used to form the following matrix:

$$\frac{d}{dt} \begin{pmatrix} e \\ e^* \\ n \end{pmatrix} = \begin{bmatrix} g[(N_0 - 1) - i\alpha \cdot N_0] - \alpha_{int} & 0 & g \cdot E_0 \cdot (1 - i\alpha) \\ 0 & g[(N_0 - 1) + i\alpha \cdot N_0] - \alpha_{int} & g \cdot E_0 \cdot (1 + i\alpha) \\ -2 \cdot g \cdot E_0 \cdot (N_0 - 1) & -2 \cdot g \cdot E_0 \cdot (N_0 - 1) & -(2gI_0 + \gamma) \end{bmatrix} \begin{pmatrix} e \\ e^* \\ n \end{pmatrix} \quad [1.17]$$

Matrix [1.17] was analyzed using the following time dependent parameter values

Variable	Description	Value
α_{int}	Cavity loss (mirror loss)	1
g	Gain coefficient	1
α	Line width enhancement factor	3
J_{pump}	Pump current	$.001 \leq J_{pump} \leq .02$
γ	Carrier lifetime (inverse time)	.001
t	Time	0

Table 1.1

The following eigenvalues were obtained through the use of Matlab corresponding to matrix [1.17] at $t = 0$ and $J_{pump} = 0.02$.

$$\begin{aligned} \lambda_1 &= -0.0090 - 6.0030i \\ \lambda_2 &= -0.0010 - 0.0000i \\ \lambda_3 &= -0.0090 + 6.0030i \end{aligned} \quad [1.18]$$

The matrix has time dependence through the terms involving $E_0(t)$. This means that the eigenvalues will not necessarily tell us about stability. We will reformulate in the

following section to make a time independent form. Nonetheless, we looked at the eigenvalues to better understand the behavior of the laser on system.

What we did in our next step in analyzing matrix [1.17] was to observe how the eigenvalues depend on different J_{pump} inputs. Selecting a range of values for J_{pump} , from 0.001 to 0.02, and graphing the results in the following graph allowed for analysis of the eigenvalues.

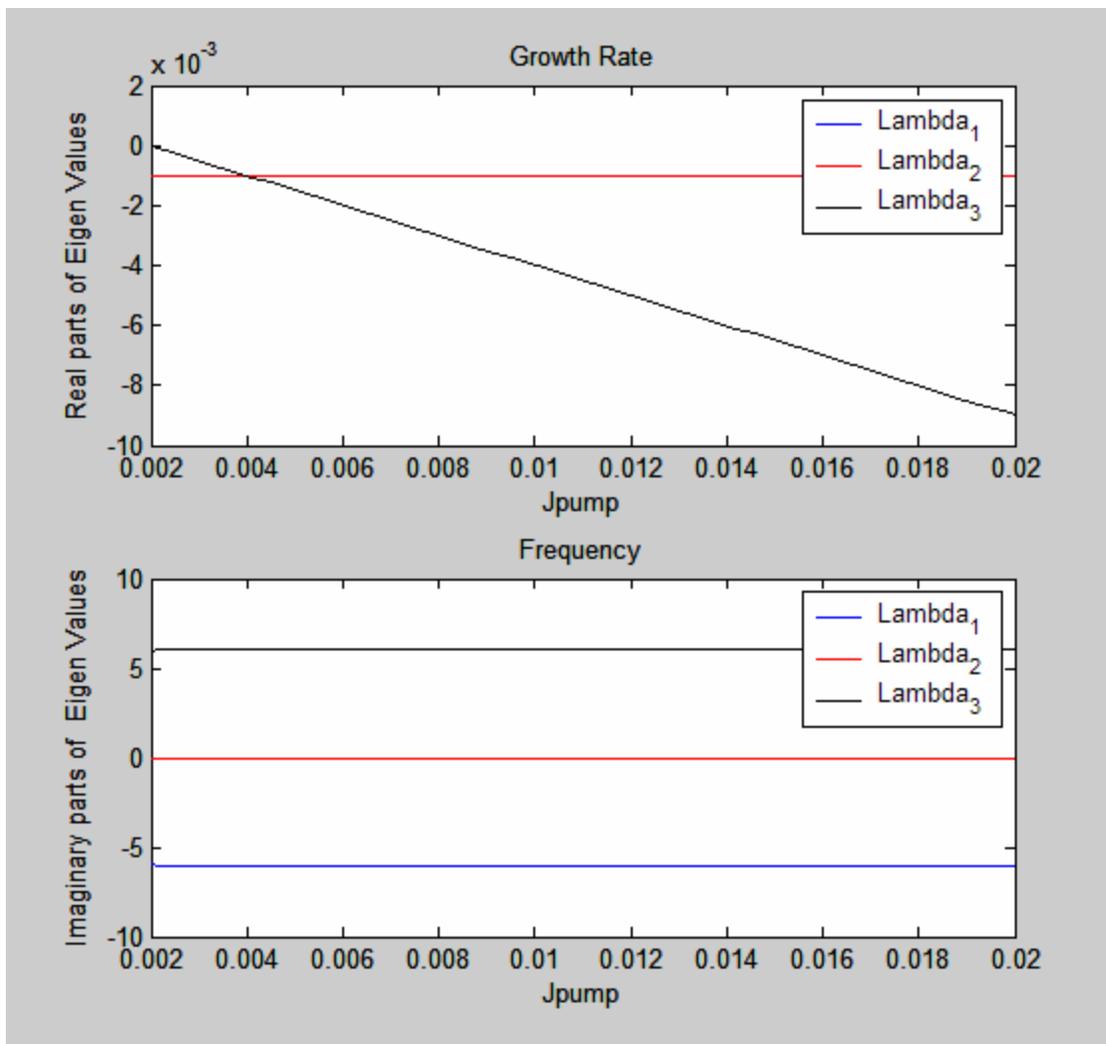


Figure 1.1

In order to find the reliability of this technique for the use of stability analysis t was varied. The resultant eigenvalue solutions lost complex conjugate relations as t was varied proving that this technique is not valid for stability analysis. The following data was taken at $t = 4$ and the eigenvalue solutions were evaluated at $J_{\text{pump}} = 0.02$.

$$\begin{aligned}\lambda_1 &= -0.0090 - 6.0030i \\ \lambda_2 &= -0.0135 + 0.0118i \\ \lambda_3 &= 0.0035 + 5.9912i\end{aligned}$$

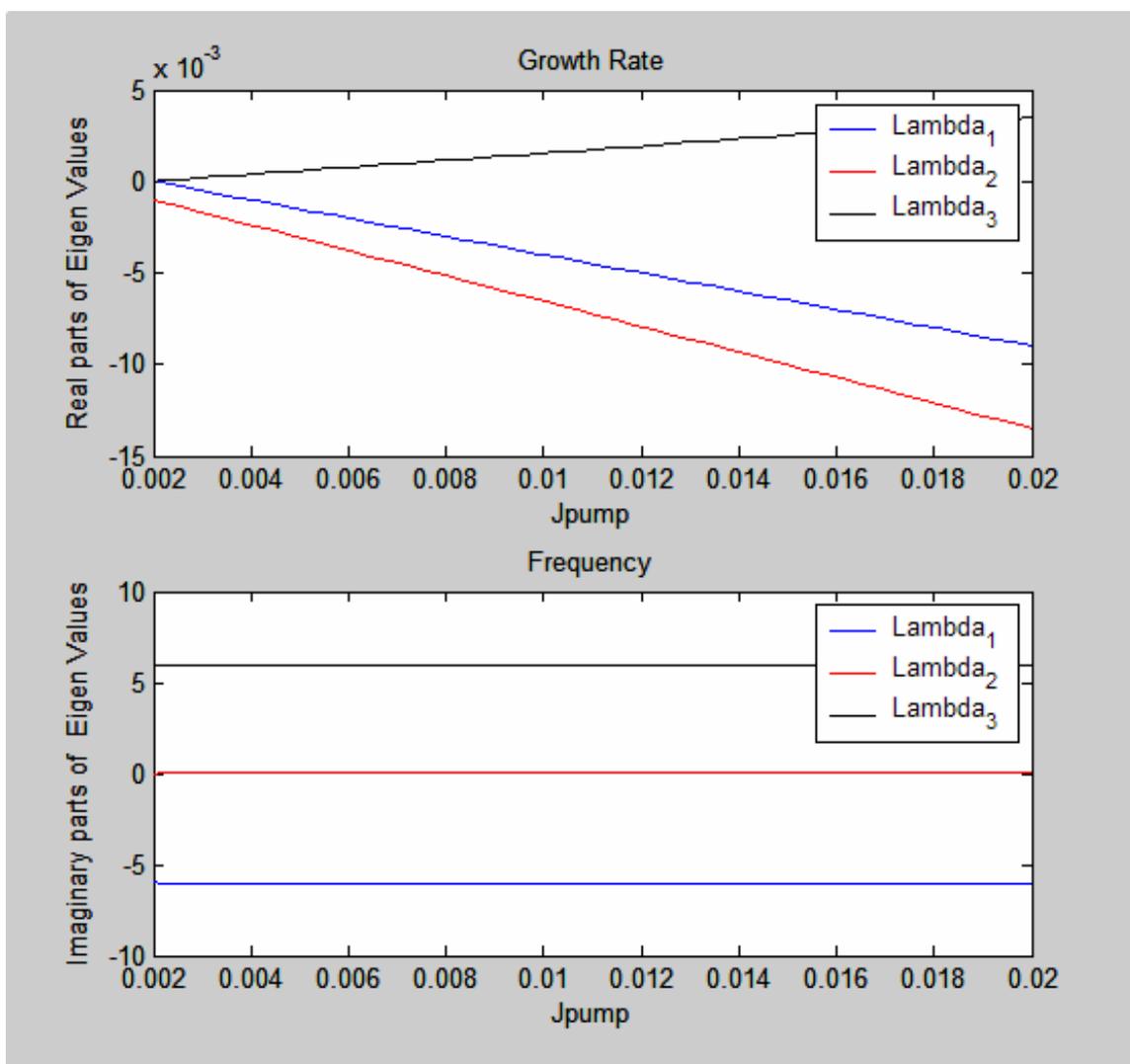


Figure 1.2

Since this technique did not allow for stability analysis, a t -independent analysis must be made.

Why do eigenvalues work to analyze stability?

We must start with a basic relationship $dv/dt = Av$, where A is a constant diagonalizable matrix and $v=v(t)$ is a vector. We need to find three eigenvalue, eigenvector pairs (λ_j, v_j) so that the v_j are linearly independent. The initial value $v(0)$ may be written as:

$$v(0) = a_1v_1 + a_2v_2 + a_3v_3$$

We may now write the solution in terms of the eigenvectors as:

$$v(t) = a_1(t)v_1 + a_2(t)v_2 + a_3(t)v_3$$

Plug $v(t)$ into the equation. v_j are independent of t :

$$dv/dt = v_1da_1(t)/dt + v_2da_2(t)/dt + v_3da_3(t)/dt$$

Therefore:

$$\begin{aligned} Av &= a_1(t)Av_1 + a_2(t)Av_2 + a_3(t)Av_3 \\ &= a_1(t)\lambda_1v_1 + a_2(t)\lambda_2v_2 + a_3(t)\lambda_3v_3 \end{aligned}$$

Since the v_j are assumed linearly independent we may now write:

$$da_j(t)/dt = \lambda_j a_j(t)$$

and because the λ_j are constant

$$a_j(t) = a_j(0)e^{\lambda_j t}.$$

$v(t)$ may now be rewritten as the follows:

$$v(t) = a_1(0)e^{\lambda_1 t} + a_2(0)e^{\lambda_2 t} + a_3(0)e^{\lambda_3 t}$$

Concerning the question of how eigenvalues work to analyze stability, we see that the signs of the real parts of the eigenvalues λ_j determine whether or not $v(t)$ grows or shrinks.

Eigenvector Time Dependence

The analysis above required both eigenvalues and eigenvectors to be time independent. If the eigenvalues depend on time, the argument can be modified and used to analyze stability. However, if the eigenvectors are time dependent the method is inapplicable. The basic equation relating the perturbed electric field envelope and the carrier density is the following:

$$\frac{d}{dt} \begin{pmatrix} e \\ e^* \\ n \end{pmatrix} = A(t) \cdot \begin{pmatrix} e \\ e^* \\ n \end{pmatrix}$$

$A(t)$ is given by [1.17] and E_0 in equation [1.17] is time dependent.

Where: $E_0(t) = \sqrt{I_0} \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t}$

In order to find the eigenvector time dependence several eigenvectors were calculated for different values of time ($t = 1, 2, \& 3$). Using Matlab the following matrices $V(t)$, whose columns are eigenvectors for $A(t)$, were calculated for each time .

$V(0)=$

0.9995	-0.0473 - 0.0158i	0.0002 - 0.0007i
0.0002 + 0.0007i	-0.0473 + 0.0158i	0.9995
-0.0000 - 0.0316i	0.9975	-0.0000 + 0.0316i

$V(1)=$

0.9995	-0.0410 - 0.0283i	0.0006 - 0.0005i
0.0002 + 0.0007i	-0.0499 + 0.0019i	0.9995
-0.0089 - 0.0303i	0.9975	-0.0089 + 0.0304i

$V(2)=$

0.9995	-0.0314 - 0.0386i	0.0008 - 0.0001i
0.0002 + 0.0007i	-0.0485 - 0.0121i	0.9995
-0.0170 - 0.0266i	0.9975	-0.0170 + 0.0267i

The next step is to determine if the quotient of two such eigenvector matrices at different times produce a product of a diagonal matrix and a permutation matrix. The following is the justification for using this method to determine t-dependence.

Time	Eigenvector		
t=0	V ₁	V ₂	V ₃
t=1	W ₁	W ₂	W ₃

If the eigenvectors are “the same” that means each V_i is proportional to a different one of the W 's.

$$\begin{aligned} V_1 &= a_1 W_{J_1} \\ V_2 &= a_2 W_{J_2} \\ V_3 &= a_3 W_{J_3} \end{aligned}$$

Where $\{J_1, J_2, J_3\} = \{1, 2, 3\}$

The eigenvector matrix for $t = 0$ is equal to:

$$V = (V_1 | V_2 | V_3) = (a_1 \cdot W_{J_1} | a_2 \cdot W_{J_2} | a_3 \cdot W_{J_3})$$

For example, if $J_1 = 1$, $J_2 = 2$, and $J_3 = 3$ then.

$$V = (V_1 | V_2 | V_3) = (a_1 \cdot W_2 | a_2 \cdot W_3 | a_3 \cdot W_1)$$

The permutation matrix is now factored.

$$V = (W_1 | W_2 | W_3) \cdot \begin{pmatrix} 0 & 0 & a_3 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}$$

Therefore;

$$W^{-1} \cdot V = \begin{pmatrix} 0 & 0 & a_3 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}$$

We will now calculate the permutation for $V(0)$, $V(1)$, and $V(3)$. If any of the quotient matrices are not a diagonal matrix times a then the eigenvectors are not constant. The following data is the result from those calculations.

$V(0)/V(1)=$

$$\begin{array}{ccc} 0.9996 - 0.0001i & -0.0000 + 0.0001i & -0.0064 + 0.0126i \\ -0.0004 + 0.0002i & 1.0004 + 0.0000i & 0.0026 + 0.0139i \\ 0.0088 - 0.0013i & 0.0088 + 0.0012i & 1.0008 + 0.0002i \end{array}$$

$V(0)/V(2)=$

$$\begin{array}{ccc} 0.9991 - 0.0000i & -0.0002 + 0.0002i & -0.0160 + 0.0229i \\ -0.0007 + 0.0005i & 1.0008 + 0.0004i & 0.0011 + 0.0280i \\ 0.0170 - 0.0049i & 0.0170 + 0.0049i & 1.0015 + 0.0009i \end{array}$$

$V(1)/V(2)=$

$$\begin{array}{ccc} 0.9996 - 0.0001i & -0.0001 + 0.0000i & -0.0096 + 0.0103i \\ -0.0004 + 0.0002i & 1.0004 + 0.0003i & -0.0014 + 0.0141i \\ 0.0081 - 0.0037i & 0.0082 + 0.0037i & 1.0008 + 0.0005i \end{array}$$

The resulting data shows the eigenvectors to be time dependent.

Removing “t” Dependence in the Perturbation Equation

This section uses substitution to remove time dependence. This relates to the frequency of the waves and how the equations are not frequency dependent. We will start with the basic perturbation for the electric field envelope [1.10] and introduce $f(t)$.

$$E = e + E_0 = e + \sqrt{I_0} \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t} = (f(t) + \sqrt{I_0}) \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t} \quad [1.19]$$

Solving for e and e^* , the following equations were derived.

$$e = f \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t} \quad [1.20]$$

$$e^* = f \cdot e^{i \cdot g \cdot \alpha \cdot N_0 \cdot t} \quad [1.21]$$

We will now find two different solutions to dE/dt in order to solve for df/dt . The first equation is:

$$\frac{d}{dt} E = \frac{d}{dt} E_0 + \frac{d}{dt} e = \frac{d}{dt} E_0 + \frac{d}{dt} f \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t} - i \cdot g \cdot \alpha \cdot N_0 \cdot f \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t} \quad [1.22]$$

Where $\frac{d}{dt} E_0$ is given by [1.1].

The second equation comes from the perturbation [1.12].

$$\begin{aligned} \frac{d}{dt} E &= -\alpha \text{int}(E_0 + e) + g \cdot [(N_0 + n - 1) - i \cdot \alpha \cdot (N_0 + n)] \cdot (E_0 + e) \\ \frac{d}{dt} E &= \frac{d}{dt} E_0 - \alpha \text{int} \cdot e + g \cdot [(N_0 - 1) - i \cdot \alpha \cdot N_0] \cdot e + g \cdot E_0 \cdot (1 - i \cdot \alpha) \cdot n \end{aligned} \quad [1.23]$$

Equations [1.22] and [1.23] are now set equal to each other. After the reduction of the two equations the following solution is derived:

$$\frac{d}{dt} f = [g \cdot (N_0 - 1) - \alpha \text{int}] \cdot f + [g \cdot \sqrt{I_0} \cdot (1 - i \cdot \alpha)] \cdot n \quad [1.24]$$

The complex conjugate of [1.24] is:

$$\frac{d}{dt} f^* = [g \cdot (N_0 - 1) - \alpha_{int}] \cdot f^* + [g \cdot \sqrt{I_0} \cdot (1 + i \cdot \alpha)] \cdot n \quad [1.25]$$

Substitute for e and e^* in equation [1.15]. After the substitutions are made we get:

$$\frac{d}{dt} n = [-2 \cdot g \cdot \sqrt{I_0} \cdot (N_0 - 1)] f + [-2 \cdot g \cdot \sqrt{I_0} \cdot (N_0 - 1)] f^* + (-2 \cdot g \cdot I_0 + \gamma) n \quad [1.26]$$

In order to find the stability of the electric field envelope and the carrier density, eigenvalues must be found and analyzed. In order to do this equations [1.24], [1.25], and [2.16] are used to form the following matrix:

$$\frac{d}{dt} \begin{pmatrix} f \\ f^* \\ n \end{pmatrix} = \begin{bmatrix} g \cdot (N_0 - 1) - \alpha_{int} & 0 & g \cdot \sqrt{I_0} (1 - i \cdot \alpha) \\ 0 & g \cdot (N_0 - 1) - \alpha_{int} & g \cdot \sqrt{I_0} (1 + i \cdot \alpha) \\ -2 \cdot g \cdot \sqrt{I_0} (N_0 - 1) & -2 \cdot g \cdot \sqrt{I_0} (N_0 - 1) & -(2 \cdot g \cdot I_0 + \gamma) \end{bmatrix} \cdot \begin{pmatrix} f \\ f^* \\ n \end{pmatrix} \quad [1.27]$$

Matrix [1.27] was analyzed using the following time-independent parameter values

Variable	Description	Value
α_{int}	Cavity loss (mirror loss)	1
g	Gain coefficient	1
α	Line width enhancement factor	3
J_{pump}	Pump current	$.001 \leq J_{pump} \leq .02$
γ	Carrier lifetime (inverse time)	.001

Table 1.2

The following eigenvalues were obtained through the use of Matlab corresponding to matrix [1.27]. at $J_{pump} = 0.02$.

$$\begin{aligned} \lambda_1 &= -0.0095 + 0.1895i \\ \lambda_2 &= -0.0000 + 0.0000i \\ \lambda_3 &= -0.0095 - 0.1895i \end{aligned} \quad [1.28]$$

The next step in analyzing matrix [1.27] was to observe how the eigenvalues depend on different J_{pump} inputs. Selecting a range of values for J_{pump} , from 0.001 to 0.02, and graphing the results in the following graph allowed for analysis of the eigenvalues.

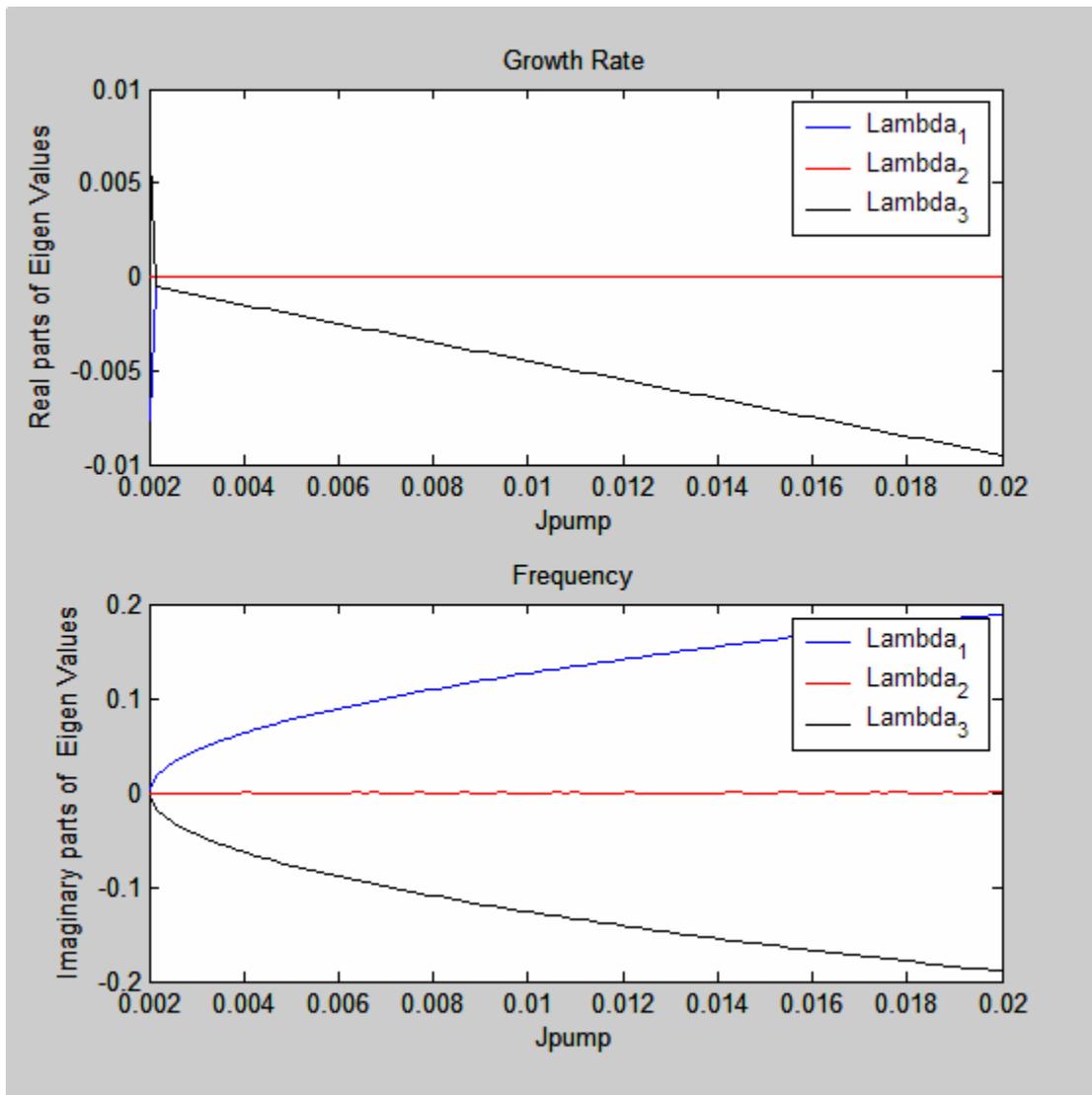


Figure 1.3 Real and imaginary parts of the eigenvalues as a function of the pumping current. note: the real parts of the first and third eigenvalues are the same.

Confirming the Results of the Stability Analysis

In order to check the result of the f, n system the solutions must be compared to the behavior of the full system. The following eigenvalue solutions to the t-independent equations were used for the continuing analysis:

Constants	Values
J _{ump}	0.0040707
λ_1	-0.0015354 + 0.064336i
λ_2	7.7977E-018 + 7.3952E-017i
λ_3	-0.0015354 - 0.064336i

Table 1.3

From equation [1.19] the following expression for f(t) was derived.

$$f(t) = (E(t) - E_0(t))e^{i \cdot g \cdot \alpha \cdot N_0 \cdot t} \quad [1.29]$$

In order to calculate E(t) a Matlab built in ode solver, Ode45, was used to solve for differential equation. We now consider initial conditions which are small perturbations of the exact lasing initial conditions $E_0(t) = \sqrt{I_0}$, $N_0(t) = N_0$:

$$\begin{aligned} E(0) &= \sqrt{I_0} + e(0) \\ N(0) &= N_0 + n(0) \end{aligned} \quad [1.30]$$

In this case we used $e(0) =$ very small value and $n(0) = 0$.

By developing [1.30], the perturbed equations may now be written as a function of t .

$$\begin{aligned} E(t) &= E_0(t) + e(t) \\ N(t) &= N_0(t) + n(t) \end{aligned} \quad [1.31]$$

Through rearranging the terms in [1.33] the following equations are formed:

$$e(t) = E(t) - E_0(t) \quad [1.32]$$

$$n(t) = N(t) - N_0 \quad [1.33]$$

We may now substitute [1.34] into [1.29] to allow for the confirmation of the solution f(t).

$$f(t) = e(t) \cdot e^{-i \cdot g \cdot \alpha \cdot N_0 \cdot t} \quad [1.34]$$

The following graphical analysis is the results from using Matlab's Ode45 to calculate $E(t)$ [1.1] and $N(t)$ [1.2] and by calculating $E_o(t)$ and $N_o(t)$. The constants values are listed in table 1.2 and 1.3, where the J_{pump} value is taken from table 1.3.

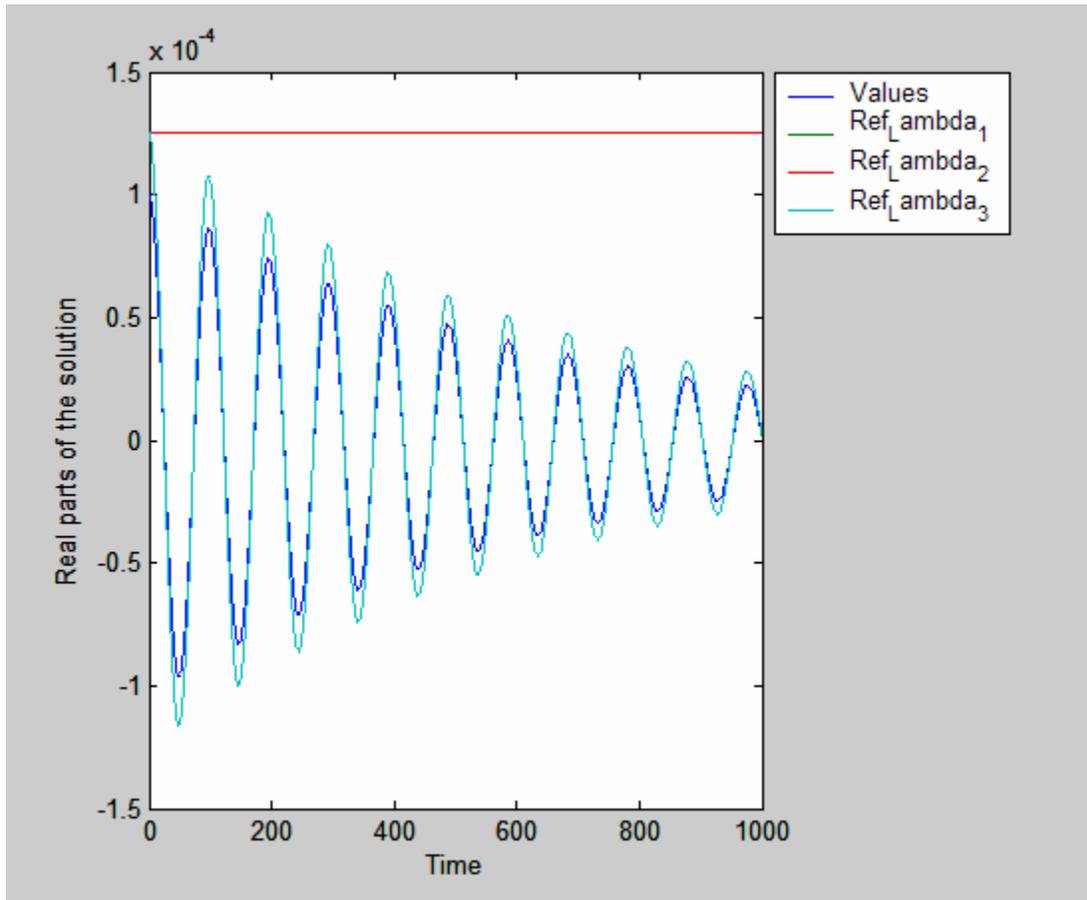


Figure 1.4

Figure 1.4 compares the calculated solution for the perturbed input to the predicted decay rate from the eigenvalue solution. From this graph we may look at the damping of the system. The matching periodicity of the perturbed input to the calculated solution is consistent between our calculated data and the predictions based on eigenvalues.

The Introduction of External Feedback

We will now add the external feedback to equation [1.1]. The external feedback added to the system depends upon the distance between the laser cavity and the mirror, the phase of the light, and the reflectivity of the mirror. As the light travels to the mirror & is then reflected back into the laser, a time delay occurs. The following term defines the external feedback for the semiconductor laser.

$$r \cdot e^{i\theta} \cdot E(t - T) \quad [2.1]$$

Variable	Description
r	Mirror reflectivity
θ	Phase (function of the type of mirror & the mirror distance)
T	Delay time

Table 2.1

Equation [1.1] may now be written as:

$$\frac{d}{dt} E = -\alpha_{int} \cdot E + g \cdot [(N - 1) - i \cdot \alpha \cdot N] \cdot E + r \cdot e^{i\theta} \cdot E(t - T) \quad [2.2]$$

Looking at Time Delay

Before calculations can be made regarding equation [2.2], an understanding of the basic qualities of a differential delay equation must be analyzed. We will start with the following simple equation.

$$\frac{d}{dt}y = -y \cdot (t - 1) \quad [2.3]$$

Where, $y(0) = 1$ & $y(t < 0) = 0$

note: the differential delay equation needs more initial data than a differential equation.

The interval of time we will analyze first is $0 \leq t \leq 1$. Both sides of equation [2.3] must be integrated, which is equal to:

$$y(t) - y(0) = - \int_0^t y(x - 1) dx$$

Through u-substitution and by plugging in $y(0) = 1$, the following equation is formed:

$$y(t) - 1 = - \int_{-1}^{t-1} y(x) dx$$

The right hand side of the equation is equal to zero because of the limit $y(t < 0) = 0$. Therefore, the solution to $y(t)$ for $0 \leq t \leq 1$ is:

$$y(t) = 1$$

The next interval for equation [2.3] to be calculated over is $1 \leq t \leq 2$. After integration, equation [2.3] may now be written as:

$$y(t) - 1 = - \int_1^t 1 dx$$

The solution of $y(t)$ for $1 \leq t \leq 2$ is:

$$y(t) = 2 - t$$

We can continue. The following table represents the first five intervals of equation [2.3] was integrated over.

Interval	$y(t) =$
$0 \leq t \leq 1$	1
$1 \leq t \leq 2$	$2-t$
$2 \leq t \leq 3$	$t^2/2 - 3t + 4$
$3 \leq t \leq 4$	$(-1/6)t^3 + 2t^2 - (15/2)t + (17/2)$
$4 \leq t \leq 5$	$(1/24)t^4 - (1/1.2)t^3 + 6t^2 - (109/6)t + (115/6)$

Table 2.2

This solution to $y(t)$ is graphed as a function of t in figure 2.1.

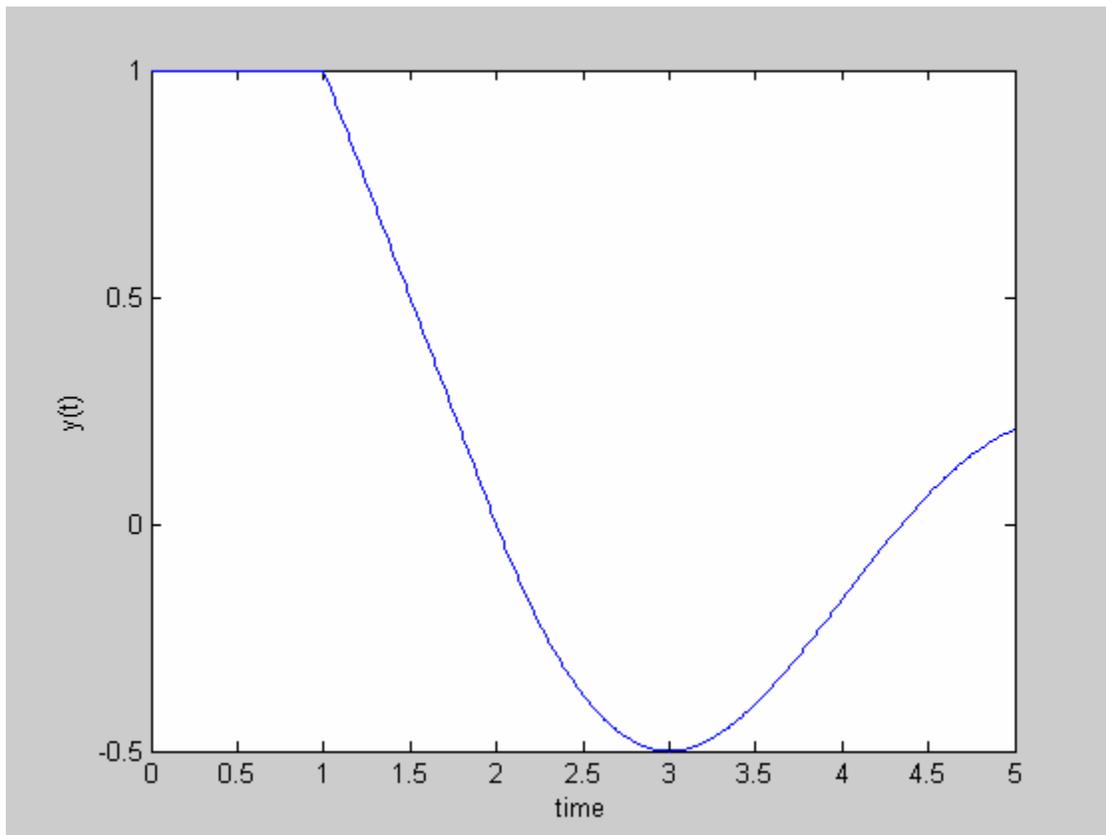


Figure 2.1

To check the solutions of the previous graph a Runge-Kutta program in Matlab was used to solve and graph the solution to $y(t)$.

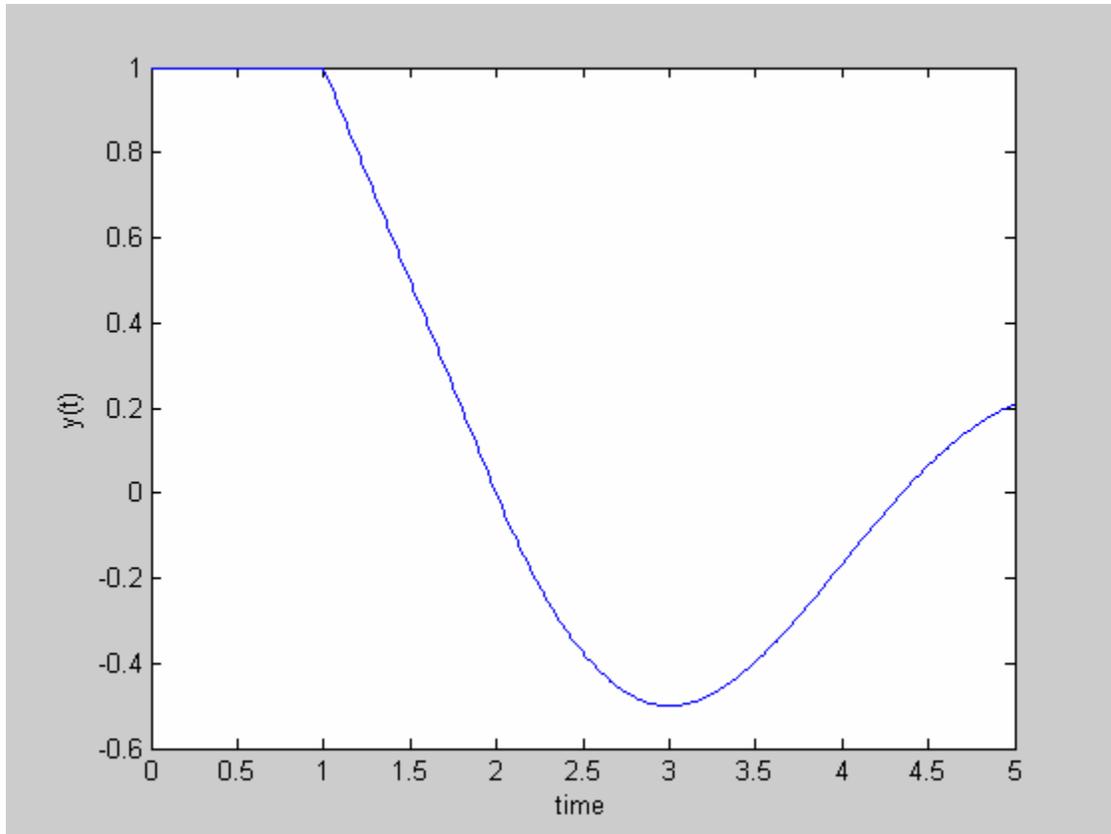


Figure 2.2

Because the graphs identically describe $y(t)$ over t . The correct values were calculated.

Equation [2.3] may now be compared to equation [2.1]. The graph for the solution to equation [2.3] is damped and therefore approaches zero, leading to a stable solution. With this feedback the solution is stable. Making the feedback strength a parameter (k) equation [2.3] becomes:

$$\frac{d}{dt}y = -k \cdot y(t - 1)$$

If $k=10$ the feedback is much stronger. The following is a graph of the solutions to equation [2.3] with a large feedback term.

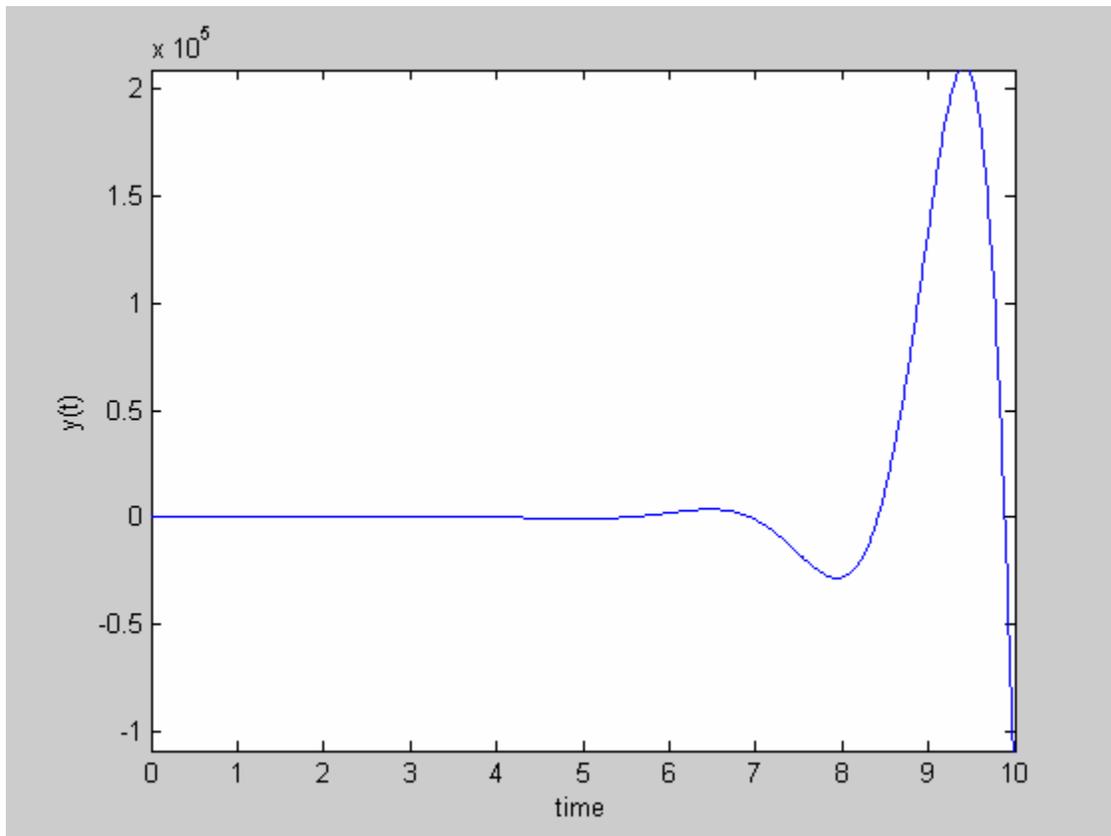


Figure 2.3

The solution is now unstable because of the increase in feedback. The increase in feedback represents a negative feedback system trying to get back to zero, but this system is incapable of reaching zero because each attempt to reach zero gets overshoot as a result of the time delay and increased weight on the system. Therefore, by observing the behavior of a simple time delay equation, we have a better understanding of how to calculate and define the stability of the semiconductor laser rate equation with the actual feedback term, equation [2.1].

Trial Solution

We will now use a trial solution to further develop the relationship between equation [2.3] and [2.1]. The trial solution will be:

$$y(t) = e^{\lambda t} \text{ for all } t \quad [2.4]$$

Where λ is complex.

Through substituting [2.4] into equation [2.3], the following equation is formed:

$$\lambda e^{\lambda t} = -e^{\lambda(t-1)}$$

This may be simplified to:

$$\lambda + e^{-\lambda} = 0 \quad [2.5]$$

Writing out λ into its complex form yields:

$$\lambda = \rho + i\sigma \quad [2.6]$$

Now substitute [2.6] into equation [2.5].

$$\rho + i\sigma + e^{-\rho} e^{-i\sigma} = 0$$

Therefore; $\rho + i\sigma + e^{-\rho} [\cos(\sigma) - i\sin(\sigma)] = 0 \quad [2.7]$

Equation [2.7] may now be broken up into real and imaginary parts.

$$\begin{aligned} \text{Real} &\rightarrow \rho + e^{-\rho} \cos(\sigma) = 0 \\ \text{Imaginary} &\rightarrow \sigma + e^{-\rho} \sin(\sigma) = 0 \end{aligned}$$

These solutions may now be graphed for their real and imaginary parts.

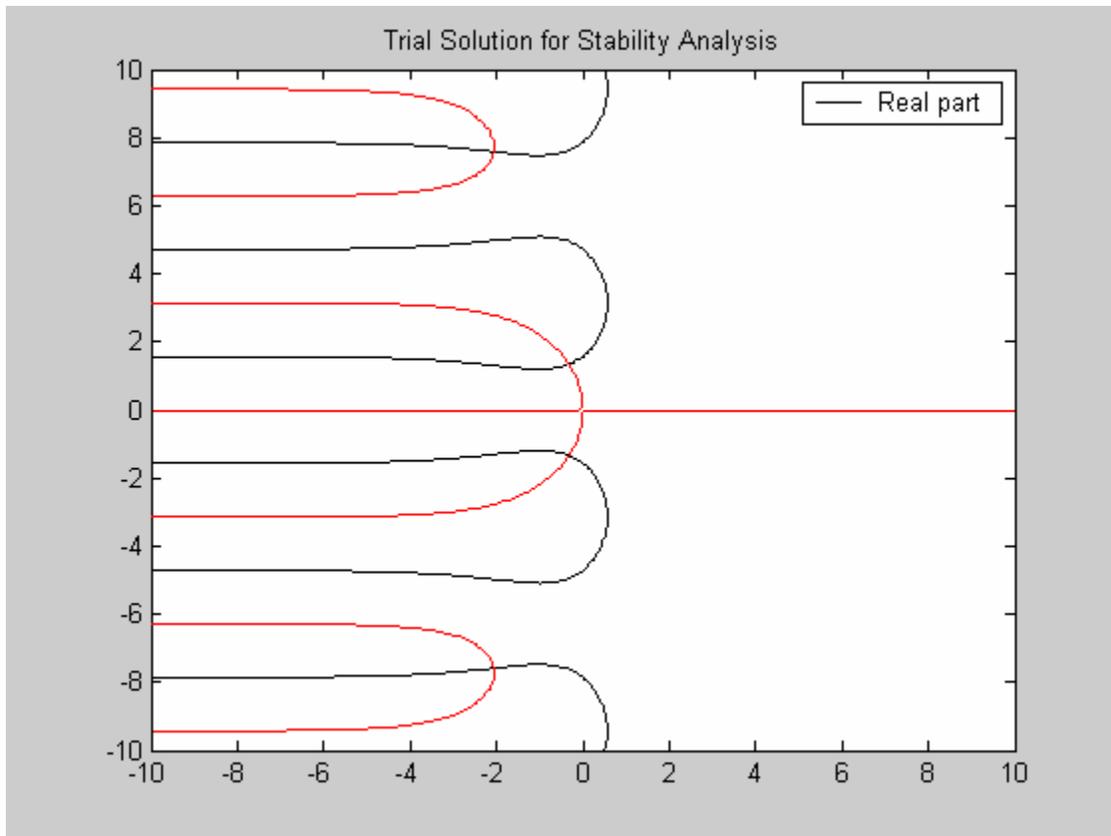


Figure 2.4

In figure 2.4, the crossing points occur for values of λ where equation [2.5] is true. Since all such crossings have negative real parts, this suggests that the equation is stable. For equation [2.4], if λ is positive, the system will diverge, leading to instability. If λ is negative the system will converge, leading to a stable solution

We will now compare the dynamics of $y' = -k y(t-1)$ for $k = 1, 1.5, 1.6,$ and 10 with the values of λ to observe the stability of the system.

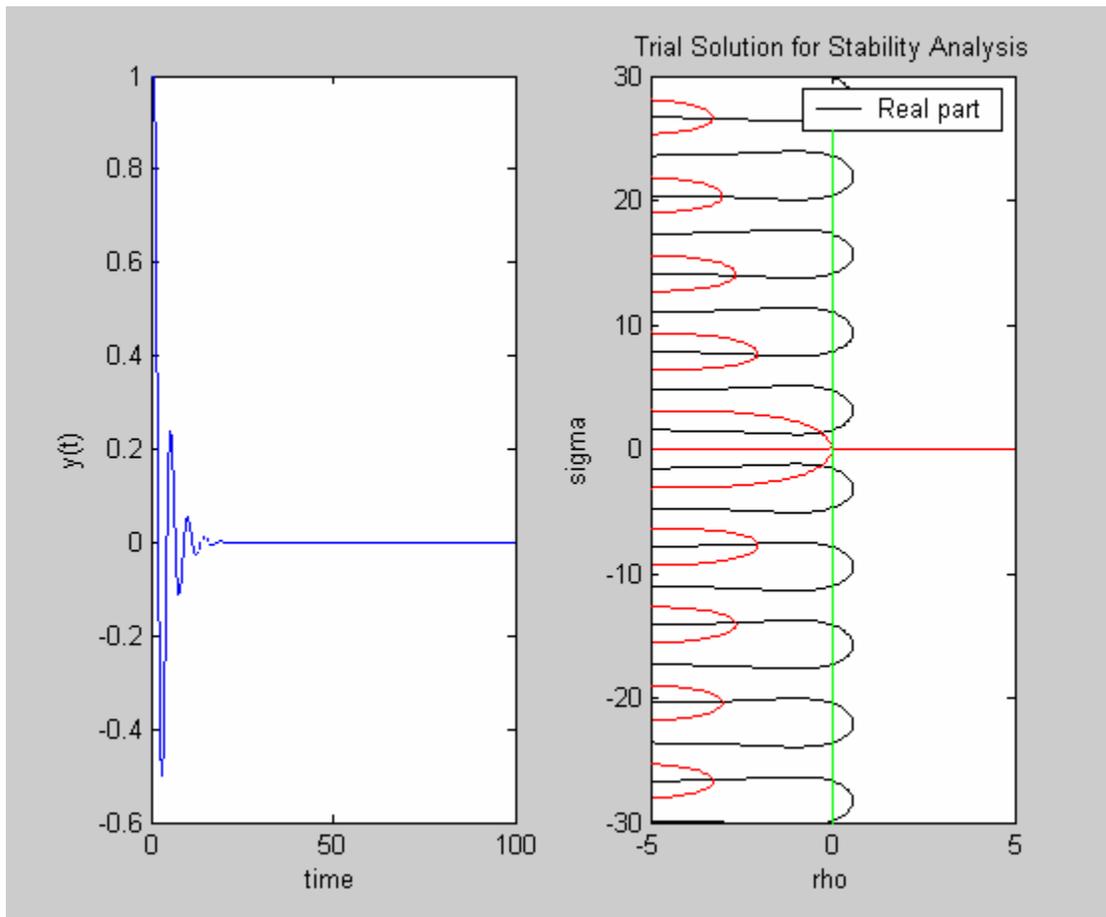


Figure 2.5 for $k = 1$.

The stable system displayed in figure 2.5 is stable because it is damped as a result of the negative intersecting λ solutions.

The following two figures display how the dynamics are greatly affected by having positive intersecting λ solutions.

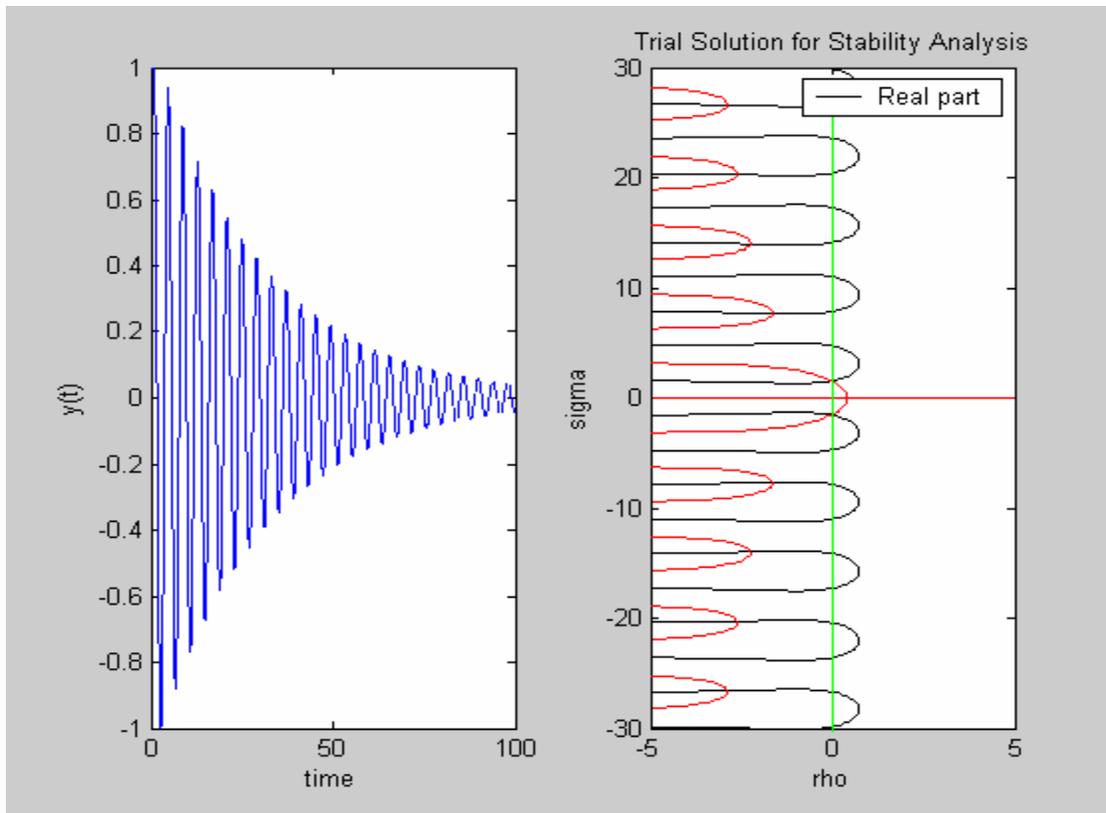


Figure 2.6 for $k = 1.5$. The system is still stable.

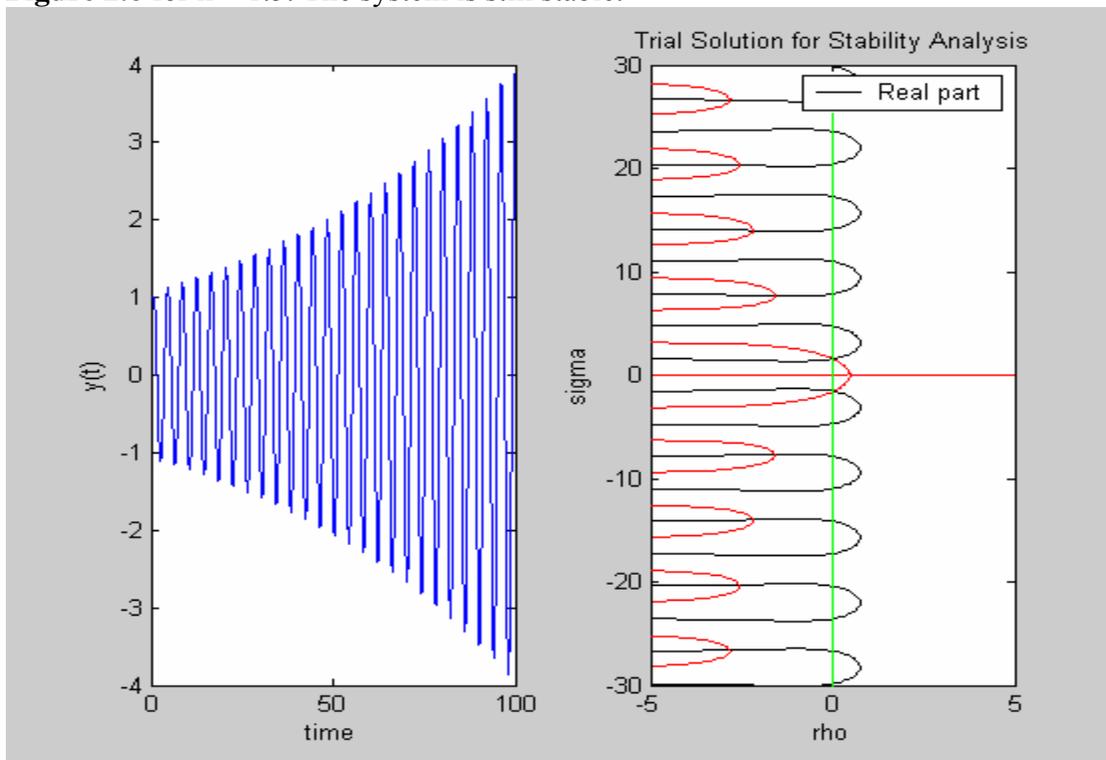


Figure 2.7 for $k = 1.6$. The system is now unstable due to solutions with a positive real part.

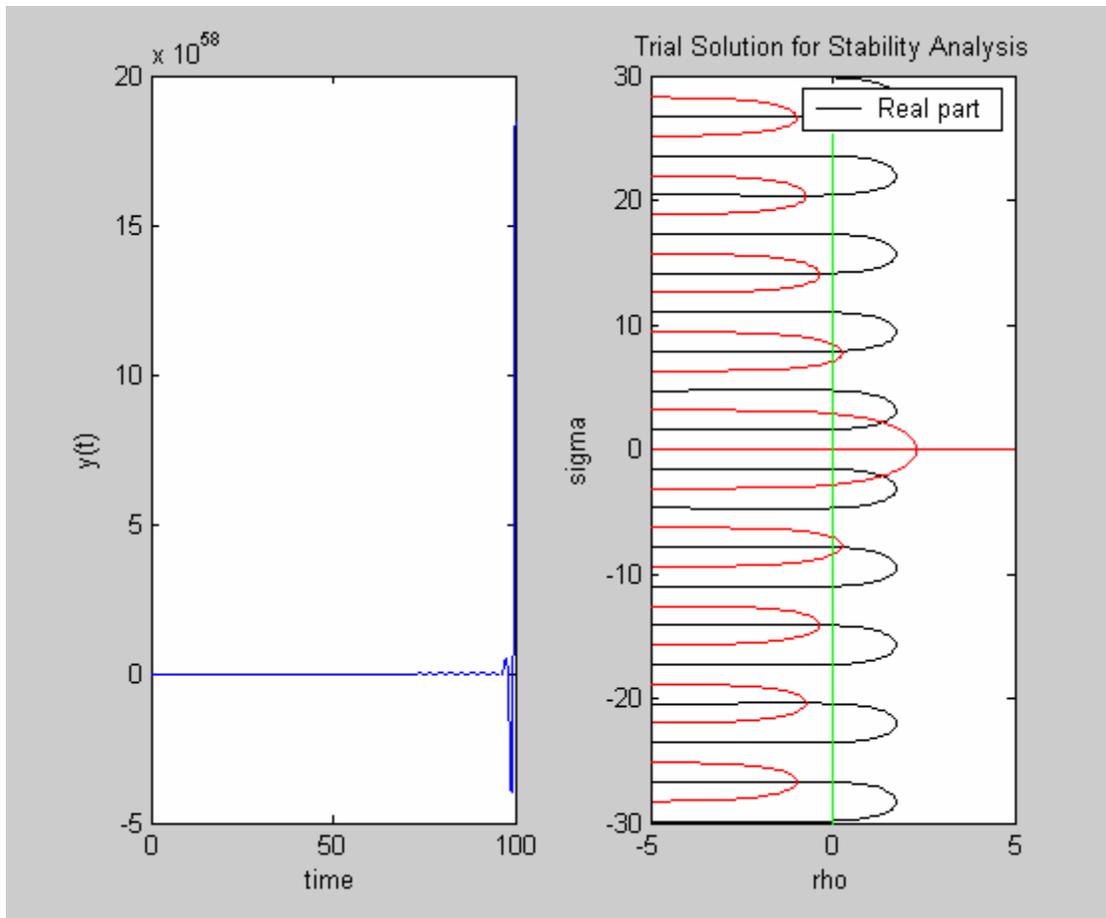


Figure 2.8 for $k = 10$.

For $k = 10$ the growth rate is much larger.

Exploring the Dynamics of the System with External Feedback

We will now return to the original system with feedback. The primary variables that will be controlled are the time delay and the reflectivity of the external mirror. By changing these variables the stability of the system is greatly affected with the introduction of the external feedback into the semiconductor laser rate equations. The following figures will make comparisons of the original semiconductor electric field envelope with and without external feedback.

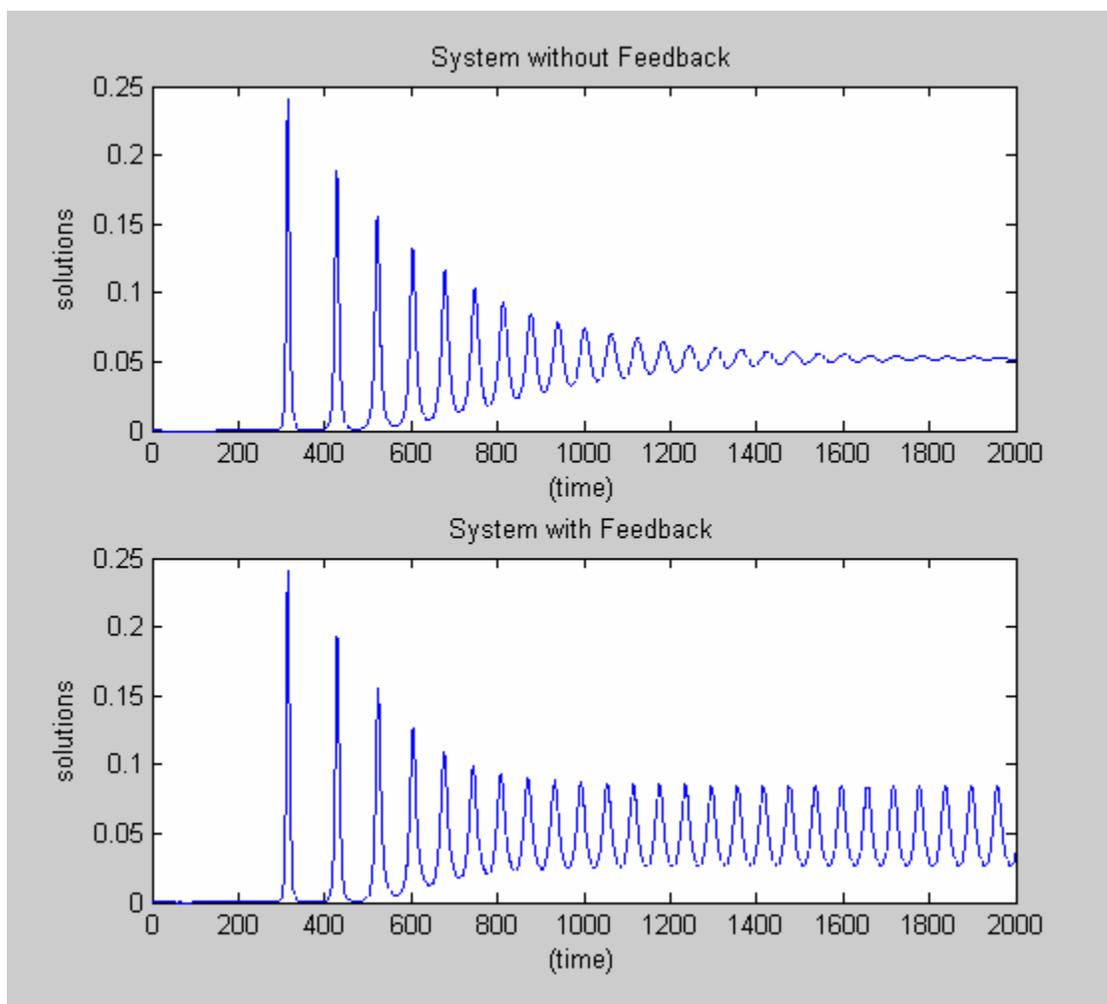


Figure 2.9 The reflectivity is at 0.01 and the time delay is set at 10.

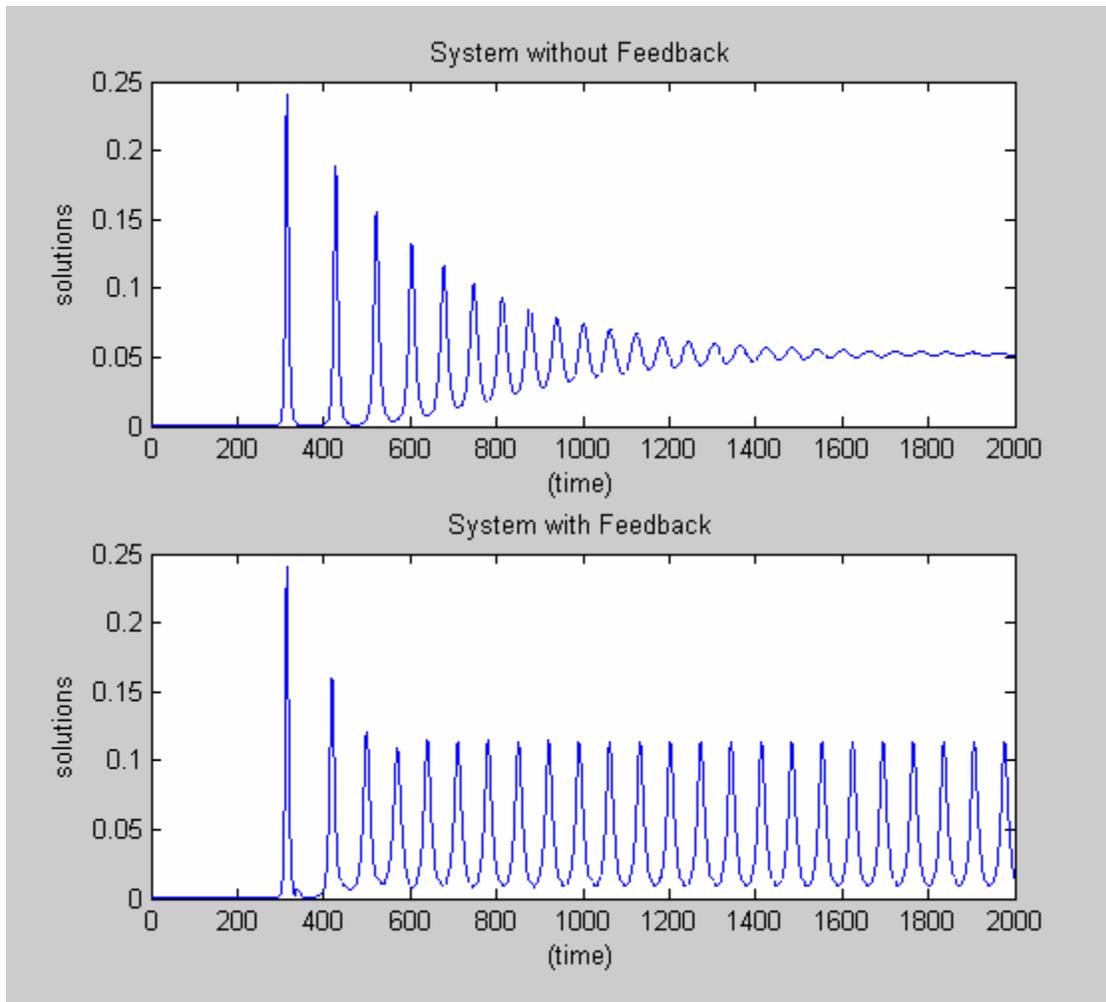


Figure 2.10 The reflectivity is at 0.01 and the time delay is set at 20.

As seen from figures 2.9 and 2.10, the external feedback equation caused the system to become unstable, which happened more rapidly in figure 2.10 than in figure 2.9 because of the increase in time delay. Both figures also display that the periodicity increases when feedback is added to the system. This occurs because of the extra energy reflected back into the system from the reflectivity of the external mirror.

Matlab codes:

The directory [Matlab Programs](#) contains the Matlab source used in this analysis.

Future Work

We have just begun to analyze the semiconductor laser system with feedback. The dynamics analysis shows that the laser has the tendency to be unstable when external feedback is added to the system. This needs to be further developed in order to make predictions for how two individual semiconductor laser systems interact when coupled together. To do this, the individual semiconductor laser must be analyzed with the feedback in the following ways.

We will first find exact solutions to the semiconductor laser rate equations with the added feedback. This will give solutions that have a dependence on both the original time (t) and the delay time (T). These solutions will be derived using a similar method to that done in the section labeled *Finding Solutions for the Laser on Case*, and will provide a starting point to begin analyzing the stability of the system.

In order to analyze the stability of the semiconductor laser system with feedback the new laser rate equations must be perturbed and analyzed. This will be done by finding eigenvector, eigenvalue pairs that will be used to determine the dependence of the system through numeric and graphical analysis. After this section of the research has been completed we will be able to couple the now known unstable laser systems.

Coupling two individual semiconductor laser systems with feedback has been the main goal of this research because of the problem involving the chaotic nature of coupled semiconductor laser systems. From the development of the laser rate equations and the respective analysis done to examine stability of the laser rate equations the coupling of two of these individual lasers will then be possible. By coupling these lasers, we will be able to make predictions regarding the chaotic nature of the coupled system and therefore solve the problem of not being able to make predictions of the laser emissions because of their chaotic nature.

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