The Non-Commutative Geometry of Penrose Tilings

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Abstract

This paper represents work completed during the 2003-2004 academic year, as well as summer 2004. It is written for a curious math undergraduate, assuming some limited background. Gelfand's Theorem is proved in detail to provide familiarity with associating C*-algebras to compact, Hausdorff spaces. Penrose tilings are then studied, concluding with an algebraic characterization of the space of Penrose tilings using the methods of algebraic K-theory.

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Contents

1 Introduction

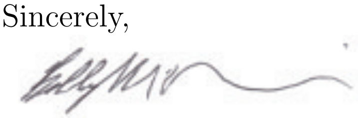
This paper was written over the course of three semesters. The writing began in the Fall of 2003, after working with Arlo Caine studying planar tilings. During this time, we studied Penrose tilings in some depth. Many of these results are presented in Section 4. By the end of the Fall semester, we decided to continue with the full scope of the project, which required that I learn some more advanced math. In the Spring of 2004, I proved the first part of Gelfand's Theorem and then in the beginning of the summer I completed the proof (Sections 2 and 3). At this point, I was ready to look at the advanced methods of non-commutative geometry to study the non-commutative space of Penrose tilings. The results of this effort are presented in sections 5-8.

This paper is written for an undergraduate mathematics student with some knowledge of algebra and analysis. First, we will study Gelfand's Theorem to see how one can associate an abelian algebra with a space. The proof is given in sufficient detail so that the intended audience, provided some standard references for unfamiliar definitions, etc., can understand every step.

Once Gelfand's theorem has been explained, we summarize the basic properties of Penrose tilings. Then using these properties, we develop a way of coding Penrose tilings so that the set of all Penrose tilings can be analyzed. We briefly attempt to use classical methods to study this space, but we find that these methods fail. Finally, we use methods from non-commutative geometry, and we see that the results are successful.

I would like to thank Arlo Caine for the many hours he has spent teaching me. His ability to condense many large subjects and communicate them to a young undergraduate reveals his mastery. I would also like to thank Dr. Doug Pickrell for overseeing the project and assisting me when Arlo could not.

Sincerely,



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2 Compact Hausdorff Spaces

Gelfand's Theorem applied to compact Hausdorff spaces is the following statement:

$$X \cong \operatorname{Spec} C(X)$$

where X is a compact, Hausdorff space and $C(X)$ denotes the abelian C^*-algebra of continuous, complex valued functions on the set, X. Gelfand's theorem also states the converse, that every unital abelian C^*-algebra is isomorphic to continuous functions on some locally compact Hausdorff space. These two statements show that there is an equivalence of categories - studying some locally compact Hausdorff space is like studying the associated abelian C^*-algebra.

We will prove the first part of Gelfand's Theorem in this section, by showing that there exists a bijective function from X onto $\operatorname{Spec} C(X)$. Then, we show that this relation is actually a homeomorphism between compact, Hausdorff spaces. First, we provide some background.

We begin by stating that $C(X)$ is a complex vector space under pointwise addition. This is well known, and can easily be verified. Now, we show that it is a complex algebra under pointwise multiplication.

We know that an algebra is a vector space, which is equipped with an associative multiplication that commutes with scalars (from the field, over which the vectorsspace is defined). Here, we define

$$\forall x \in X \qquad (f \cdot g)(x) = f(x) \cdot g(x)$$

and all of the conditions are clearly satisfied. It can also be pointed out that this algebra is abelian, since it inherits commutivity from multiplication on $\mathbb{C}$.

We go further, and verify that $C(X)$ is actually a C^*-algebra, by showing that $\forall f, g \in C(X)$,$\forall a \in \mathbb{C}$, $\exists\ ^* : A \to A$ such that

$$(f^*)^* = f \qquad (f+g)^* = f^* + g^* \qquad (f \cdot g)^* = g^* \cdot f^* \qquad (a \cdot f)^* = \bar{a} \cdot f^*$$

These identities are obvious when we recognize that each element of $C(X)$ can be decomposed: $f(x) = f_1(x) + i \cdot f_2(x)$, where f_1, f_2 are real valued. Then, the $^*-$operation is simply complex conjugation.

Finally, we verify that $C(X)$ is a unital C^*-algebra, by showing that there exists an element, $f \in C(X) \ni \forall g \in C(X)$, $\forall x \in X$, $(f \cdot g)(x) = g(x)$. This is the element $f \ni$

$$\forall x \in X \qquad f(x) = 1$$

Now that we have $C(X)$ as a unital C^*-algebra, we would like to define a norm on $C(X)$, such that $C(X)$ is complete with respect to the metric induced from the norm.

Define $\parallel \parallel: C(X) \to \mathbb{R}$ such that

$$\parallel f \parallel = \max_{x \in X} |f(x)|$$

Clearly, this is a norm provided that the max exists. Since X is a compact topological space, and $\parallel f \parallel$ is a continuous function from X into $\mathbb{R}$, we know that the function attains its maximum value.

We note that $C(X)$ is complete with respect to this norm, though the proof is omitted.

Now, we're ready to consider the relation between the set, X, and the set, $\mathrm{Spec}\, C(X)$. $\mathrm{Spec}\, C(X)$ is defined to be the set of all non-zero C^*-homomorphisms that map elements of $C(X) \to \mathbb{C}$. We will now show that there is an injective correspondence between points of these sets.

We will actually prove that every non-zero C^*-homomorphism, which we will call ϕ, returns the value $f(x)$ for some point $x \in X$. Then the injective relationship between X and the set of all ϕ's is obvious: for every x, there is a ϕ_x such that $\forall f \in C(X)$, $\phi_x(f) = f(x)$.

If $\phi : C(X) \to \mathbb{C}$ is a non-zero, C^*-homomorphism, then $\exists\, x \in X \ni \forall\, f \in C(X)$, $\phi_x(f) = f(x)$.

Proof: We proceed by contradiction. Assume, then, $\forall x \in X, \exists f \in C(X) \ni \phi_x(f) \neq f(x)$. Since $\phi : C(X) \to \mathbb{C}$ sends an infinite dimensional space to a one dimensional space, we know that the kernel of ϕ, the set $\{f \in C(X) | \phi(f) = 0\}$, is non-empty. Consider an $x \in X$. By our assumption, $\exists f_x \ni \phi(f_x) = 0$ and $f_x(x) \neq 0$. Since the kernel of a homomorphism is an ideal, we note that f_x is in an ideal.

Now consider $\bar{f}_x$. Since ϕ is a C^*-homomorphism, it is clear that $\phi(\bar{f}_x) = \phi(f_x)^* = 0^* = 0$, so $\bar{f}_x$ is also in this ideal. Now, since $f_x \neq 0$, it must be that $\bar{f}_x \neq 0$. Now, we have the non-zero product $f_x \cdot \bar{f}_x = |f|^2$, which is in $ker(\phi)$ (since it is an ideal). Now, we will use the compactness of X to show that $1 \in ker(\phi)$.

For each $x \in X$, we have shown that we can find a continuous function $|f_x|^2$, that is real-valued and non-zero in some neighborhood about x. The union of all such neighborhoods is an open cover of the space, X. Then, since X is compact, we know that there exists a finite subcover, which covers X. Consider this finite subcover. We have a finite set of functions, $\{f_{x_1}, \ldots, f_{x_n}\}$. Let $f = |f_{x_1}|^2 + \ldots + |f_{x_n}|^2$. Since ϕ is linear, it must be that $\phi(f) = \phi(|f_{x_1}|^2) + \ldots + \phi(|f_{x_n}|^2) = 0$, so $f \in ker(\phi)$, and f is continuous and non-zero over all of X.

Now, consider $(f(x))^{-1}$. Since f is continuous and non-zero, so is $(f(x))^{-1}$. Now, $ker(\phi)$ must contain the multiplicative identity, since $f \cdot (f(x))^{-1}$ must be in the ideal, by definition. This means that $ker(\phi) = C(X)$, since any ideal containing the algebra's identity must be the entire algebra. But this implies that ϕ is a zero homomorphism, contradicting our assumption. We conclude that every non-zero C^*-homomorphism must be evaluation at a point for some point $x \in X$. □

At this stage in the proof, we have demonstrated a one-to-one correspondence between the compact Hausdorff space, X, and the set of all C^*-homomorphisms, $\operatorname{Spec} C(X)$. Now, we would like to demonstrate that there is a homeomorphism between these compact Hausdorff spaces. Of course, this requires that we impose a canonical Hausdorff topology on $\operatorname{Spec} C(X)$.

We notice that $\operatorname{Spec} C(X)$ is actually a subset of a much larger space, $C(X)^\vee$. This is called the *dual* space of $C(X)$, and is defined as the set of *all* continuous, linear functionals mapping the algebra to its base field (in this case $\mathbb{C}$). Clearly, this differs from $\operatorname{Spec} C(X)$, since it lacks the strict requirement that these functionals be multiplicative homomorphisms. There is a natural topology to put on this larger space, called the weak$-^*$ topology. This topology is generated by sets of the form:

$$N(\phi_0 : S, \epsilon) := \{\phi \in C(X)^\vee : |\phi(f) - \phi_0(f)| < \epsilon \; \forall f \in S \subset C(X)\}$$

where S must be a finite subset. We note explicitly that each basis element depends on three parameters: ϕ_0, ϵ, S. We would like to show that the following:

The weak$-^*$ topology is Hausdorff.

Proof: To show that this topology is Hausdorff, we consider $\phi_1, \phi_2 \in C(X)^\vee \ni \phi_1 \neq \phi_2$. We want to show that there are two neighborhoods containing ϕ_1, ϕ_2, respectively, that are disjoint. We proceed constructively, by finding neighborhoods which meet these requirements.

First, we define the parameters. Let $N_1 = N(\phi_1 : \{g\}, \epsilon)$, and let $N_2 = N(\phi_2 : \{g\}, \epsilon)$ for any $g \ni \phi_1(g) \neq \phi_2(g)$ (Such a g must exist since $\phi_1 \neq \phi_2$). Then, let $\epsilon = \frac{|\phi_1(g) - \phi_2(g)|}{2}$.

Thus, we have the following neighborhoods:

$$N_1 = \{\phi \in C(X)^\vee : |\phi(g) - \phi_1(g)| < \frac{|\phi_1(g) - \phi_2(g)|}{2}\}$$

$$N_2 = \{\phi \in C(X)^\vee : |\phi(g) - \phi_2(g)| < \frac{|\phi_1(g) - \phi_2(g)|}{2}\}$$

By our choice of g, it is clear that both neighborhoods are non-empty, for they must contain ϕ_1, ϕ_2 respectively. Furthermore, these neighborhoods have null intersection. □

Now, we would like to define a norm on $C(X)^\vee$. Consider any $\phi \in C(X)^\vee$. It can be verified that there is a bound on $\frac{|\phi(f)|}{\|f\|}$ that is independent of $f \in C(X)$. Then, we define $\| \|: C(X)^\vee \to \mathbb{R}$ as

$$\| \phi \| = \sup_{f \neq 0} \frac{|\phi(f)|}{\| f \|}$$

$\| \|$ *defines a norm on* $C(X)^\vee$.

Proof: We verify that this definition satisfies the conditions. $\frac{|\phi(f)|}{\|f\|} > 0 \; \forall f \neq 0 \in C(X)$ by definition of absolute value and the norm. It follows that this supremum must also be positive. Now, consider $\alpha \in \mathbb{C}$. Then, clearly $\| \alpha \cdot \phi \| = |\alpha| \cdot \| \phi \|$, by properties of ϕ and suprema. Finally, since suprema respect the triangle inequality, we have that $\forall \phi_1, \phi_2 \in C(X)$,

$$\| \phi_1 + \phi_2 \| \leq \| \phi_1 \| + \| \phi_2 \|.$$ □

In order to prove that $\operatorname{Spec} C(X)$ is a *compact* Hausdorff space, we will need to show that it is a closed subset of the unit ball, which is compact in the weak-* topology by the Banach-Alaoglu Theorem.

$\operatorname{Spec} C(X)$ *is bounded by the unit ball in* $C(X)^\vee$.

Proof: We consider an arbitrary $\phi_x \in \operatorname{Spec} C(X)$. Then ϕ is given by evaluation at x. We want to show that the norm of ϕ is ≤ 1.

Now, $\| \phi_x \| = \sup_{f \neq 0} \frac{|\phi_x(f)|}{\|f\|}$, and we defined $\| f \| = \max_{x \in X} |f(x)|$. Combining these expressions, we have that

$$\| \phi_x \| = \sup_{f \neq 0} \frac{|\phi_x(f)|}{\max_{x \in X} |f(x)|}$$

Clearly, $\| \phi_x \| \leq 1$ iff $\frac{|\phi_x(f)|}{\max_{x \in X} |f(x)|} \leq 1 \forall f \in C(X)$. But this result is obvious, because $|\phi_x(f)| = |f(x)| \leq \max_{x \in X} |f(x)|$ by definition of maximum. Notice that we can easily verify that $\forall x \in X$, $\|\phi_x\| = 1$. We only need to check that there exists an $f \in C(X)$ that attains its maximum value at x, since this would lead to $\frac{\max_{x \in X} |f(x)|}{\max_{x \in X} |f(x)|} = 1$. Clearly, the unital $(f(x) = 1, \forall x)$ satisfies this property. □

$\operatorname{Spec} C(X)$ *is closed in the weak***−topology.*

Proof: To show that $\operatorname{Spec} C(X)$ is closed, we must show that its complement is open in the weak*−topology. Thus, we want to construct an open set containing

$C(X)^\vee \setminus \operatorname{Spec} C(X)$. Clearly, this can be done if we can find an open set around an arbitrary point in $C(X)^\vee \setminus \operatorname{Spec} C(X)$ that is disjoint from $\operatorname{Spec} C(X)$.

We consider an element, $\phi \in C(X)^\vee \setminus \operatorname{Spec} C(X)$, where $\phi \neq 0$. Since $\phi \in C(X)^\vee$, ϕ is a continuous, linear functional from $C(X) \to \mathbb{C}$. In particular, ϕ is not in $\operatorname{Spec} C(X)$, which means that it cannot be multiplicative. So, there must exist some elements, $f, g \in C(X) \ni \phi(f) \cdot \phi(g) \neq \phi(fg)$. Also, since the elements of $C(X)^\vee$ are continuous and linear, they must be bounded. Thus, there exists some $k \in \mathbb{R} \ni \forall\, w \in C(X)^\vee, \forall\, f \in C(X), |w(f)| \leq k\|f\|$.

Consider the neighborhood $N(\phi : \{f, g, f \cdot g\}, \frac{|\phi(f \cdot g) - \phi(f) \cdot \phi(g)|}{1 + k(\|f\| + \|g\|)})$. One can show that this neighborhood contains ϕ and no element of $\operatorname{Spec} C(X)$. □

Because $\operatorname{Spec} C(X)$ is bounded by the unit ball, and is closed, $\operatorname{Spec} C(X)$ is compact by the Banach-Alaoglu Theorem. Thus, we have established that $\operatorname{Spec} C(X)$ is a compact, Hausdorff space. Now, we want to find a homeomorpishm between X and $\operatorname{Spec} C(X)$ to complete this part of Gelfand's Theorem.

Define $\Phi : \operatorname{Spec} C(X) \to X$ by $\Phi(\phi_x) = x$. We will show that this is a continuous bijection between compact, Hausdorff spaces. We will then show that such a function is a homeomorphism. First, we must verify the following:

Φ *is continuous.*

Proof: To show that Φ is continuous, we will show that the pre-image of any closed set in X is closed in $\operatorname{Spec} C(X)$. So, we consider a closed set, $C \in X$. We will show that $\Phi^{-1}(C)$ is closed, by showing that its complement inside $\operatorname{Spec} C(X)$ is open. So, specifically we want $\{\phi_x \in \operatorname{Spec} C(X) | \Phi(\phi_x) \notin C\}$ to be an open set. This corresponds to the set$\{y \in X | y \notin C\}$. From Uryshon's lemma, define a function $f_y : X \to \mathbb{R}$ such that $f_y(y) = 0$ and $f_y(z) = 1$, $\forall z \in C$. So we consider a neighborhood

$$N(\phi_y : f_y, \frac{1}{2}) = \{w \in \operatorname{Spec} C(X) : |w(f_y) - \phi_y(f_y) < \frac{1}{2}\}$$

Then, using our definition of f_y,

$$N(\phi_y : f_y, \frac{1}{2}) = \{w \in \operatorname{Spec} C(X) : |w(f_y)| < \frac{1}{2}\}$$

So we have shown that the complement of $\Phi^{-1}(C)$ is closed. It follows that Φ is continuous. □

Φ *is a Homeomorphism between compact Hausdorff spaces.*

Proof: To prove this result, we must show that Φ is bijective, continous, and that Φ^{-1} is continuous. Since we have already shown that the first two conditions are satisfied, it remains to show that Φ^{-1} is continuous.

Consider a closed subset $A \subseteq \operatorname{Spec} C(X)$. Then, A is compact since it is a closed subset of a compact space. $\Phi(A)$ must be compact because it is the image of a compact space under a continuous map. Finally, $\Phi(A)$ is closed in X since every compact subspace of a Hausdorff space is closed.

So we have established that every closed subset in the domain space is closed in the image space. This verifies that Φ^{-1} is continuous. It immediately follows that Φ is a homeomorphism between compact, Hausdorff spaces. □

This proves that $X \cong \operatorname{Spec} C(X)$, and completes Part I.

3 Abelian C*-Algebras

The second half of Gelfand's Theorem says:

$$A \cong C(SpecA)$$

where, A is any abelian unital C^*-algebra.

Now, we should recall the spectrum of an element of a C^*-algebra; If $A^\times$ denotes the set of invertible elements of A, then the spectrum of some $x \in A$ is

$$\sigma(x) := \{\lambda \in \mathbb{C} | \lambda \cdot 1_A - x \notin A^\times\}$$

We can say that the spectrum of an element is the complement of the resolvent set. Remember, the spectrum of the algebra is *not* the collection of $\sigma(x)$ for all $x \in A$, but is the set of non-zero $^*-$homomorphisms from $A \to \mathbb{C}$.

This part of Gelfand's Theorem says that every abelian C^*- algebra looks like the algebra of continuous, complex valued functions on some compact Hausdorff space. To prove this result, we will show the existence of a bijective function from A onto $C(SpecA)$ called the Gelfand Transform, which preserves complex conjugation and all algebraic structure. This demonstrates that the Gelfand Transform is the required $^*-$isomorphism.

The Gelfand Transform

$$\hat{}: A \to C(SpecA) \quad \forall x \in A, x \mapsto \hat{x} \quad \hat{x} : SpecA \to \mathbb{C} \quad \forall \ell \in SpecA, \hat{x}(\ell) := \ell(x)$$

We would like that $\hat{}$ be a bijective homomorphism, as this would establish the isomorphism. First, we will prove that $\hat{}$ is injective. To do this, we will show that $\hat{}$ is an isometry. We begin by showing that

For every $x \in A$, range$(\hat{x}) = \sigma(x)$

Proof: Fix $x \in A$. Consider $y \in \text{Ran}(\hat{x})$. Then, $\exists\, \ell \in SpecA \ni \ell(x) = y$. We want to show that $\ell(x) \in \sigma(x)$. This means that we want $\ell(x) \cdot 1_A - x \notin A^\times$. We proceed by contradiction.

Suppose $\ell(x) \cdot 1_A - x \in A^\times$. This means $\exists\, a \in A \ni a = (\ell(x) \cdot 1_A - x)^{-1}$. We rewrite this as

$$a = \frac{1_A}{\ell(x) \cdot 1_A - x} = \frac{1}{\ell(x)} \cdot \frac{1_A}{1_A - \frac{x}{\ell(x)}} = \frac{1}{\ell(x)} \cdot \sum_{k=0}^{\infty} \left(\frac{x}{\ell(x)}\right)^k$$

Note that this representation in A is unique since the inverse of an element of a C^*-algebra is unique. However, this some can only converge in A if $\| \frac{x}{\ell(x)} \| < 1$.

This means, that $\frac{1}{|\ell(x)|} \parallel x \parallel < 1$ and so $\parallel x \parallel < |\ell(x)|$. But, this contradicts what we know about C^*-homomorphisms, namely that they are contractive (which means, $\forall x \in A, |\ell(x)| \leq \|x\|$). Therefore, a does not exist, and so $\ell(x) \cdot 1_A - x \notin A^\times$. This proves that $\mathrm{Ran}(\hat{x}) \subseteq \sigma(x)$.

Now we prove the reverse inclusion. Consider $\lambda \in \sigma(x)$. We want to show that there exists an $\ell \in SpecA$ such that $\ell(x) = \lambda$. By definition of the spectrum, we see that $\lambda \cdot 1_A - x$ is not invertible. Notice that this element generates an ideal, I, of A:

$$I = \{a \cdot (\lambda \cdot 1_A - x) | a \in A\}$$

This cannot be all of A becasue it cannot contain the element 1_A. Thus, using the Axiom of Choice we can find some maximal ideal which contains I. Every maximal ideal is the kernel of some $\ell \in SpecA$, so for this ℓ, $\ell(\lambda \cdot 1_A - x) = 0$. Then, since ℓ is a homomorphism, $\ell(\lambda \cdot 1_A - x) = \lambda \cdot \ell(1_A) - \ell(x) = 0 \Rightarrow \lambda = \ell(x) = \hat{x}(\ell)$. So, $\lambda \in \mathrm{Ran}(\hat{x})$, and it follows that $\mathrm{Ran}(\hat{x}) = \sigma(x)$. $\square$

We now would like to show that $\hat{}$ is a homomorphism of $\mathbb{C}-$algebras. A homomorphism preserves algebraic structure, so we would like to show that $\forall x, y \in A, \forall \lambda \in \mathbb{C}$

$$\widehat{x \cdot y} = \hat{x} \cdot \hat{y}$$

$$\lambda \cdot \widehat{x + y} = \lambda \hat{x} + \hat{y}$$

Since these functions share the same domain ($SpecA$), we proceed by considering the action on $\ell \in SpecA$. $\widehat{x \cdot y}(\ell) = \ell(x \cdot y) = \ell(x) \cdot \ell(y)$, since ℓ is multiplicative. Also, $\ell(x) \cdot \ell(y) := \hat{x}(\ell) \cdot \hat{y}(\ell)$. Thus, for all ℓ in the domain, $\widehat{x \cdot y} = \hat{x} \cdot \hat{y}$.

Similarly, $\hat{}$ inherits linearity from that of $\ell \in SpecA$. So $\hat{}$ is a homomorphism.

We will show that $\hat{}$ is a bounded operator. Since it is linear, this will also imply that $\hat{}$ is continuous.

$\|\hat{x}\|_\infty \leq \|x\|$, $\forall x \in A$

Proof: Fix $x \in A$. Then, $\|\hat{x}\|_\infty := \max_{\ell \in SpecA} |\hat{x}(\ell)| = \max_{\ell \in SpecA} |\ell(x)| \leq \|x\|$, since $\forall \ell \in SpecA$, $|\ell(x)| \leq \|x\|$ (ℓ is contractive). So, $\|\hat{x}\|_\infty \leq \|x\|$. $\square$

Let us prove that $\hat{}$ is a $^*-$homomorphism. This means, we want to show that $\forall x \in A$, $\overline{\hat{x}} = \widehat{x^*}$. However, we first define what is meant by $\overline{\hat{x}}$. For $\ell \in SpecA$, let $\overline{\hat{x}}(\ell) = \overline{\hat{x}(\ell)}$.

Proof: Fix $x \in A$. Consider $\ell \in SpecA$. $\widehat{x^*}(\ell) = \ell(x^*) = \ell(x)^*$, since ℓ is a $^*-$homomorphism. This means, $\widehat{x^*}(\ell) = \overline{\hat{x}(\ell)} = \overline{\hat{x}}(\ell)$, by definition. $\square$

Definition: *We define the spectral radius of $x \in A$ as*

$$r(x) := \sup_{\lambda \in \sigma(x)} |\lambda|$$

It is a theorem of Kato that $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ always exists and is equal to $r(x)$.

If x is self-adjoint, then $r(x) = \|x\|$.

Proof: We will establish that $r(x) = \|x\|$ by showing that $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \|x\|$. Since $\forall\, n \in \mathbb{N},\ \exists\, m \in \mathbb{N} \ni 2^m > n$, it suffices to show that $\forall m \in \mathbb{N},\ \|x^{2^m}\|^{\frac{1}{2^m}} = \|x\|$. We proceed by induction.

Let $m = 1$. Starting from the left, we have $\|x^2\|^{\frac{1}{2}} = \|x^*x\|^{\frac{1}{2}}$ since $x = x^*$ (x is self-adjoint). Then, by the C^*-identity, $\|x^*x\|^{\frac{1}{2}} = (\|x\|^2)^{\frac{1}{2}} = \|x\|$.

Now, let $m = k+1$. Starting from the left, we have

$$\|x^{2^{k+1}}\|^{\frac{1}{2^{k+1}}} = \|(x^{2^k})^2\|^{\frac{1}{2^{k+1}}} = \|(x^*x)^{2^k}\|^{\frac{1}{2^{k+1}}} = \|(x)^{2^k}(x)^{2^k}\|^{\frac{1}{2^{k+1}}}$$

Then, using that $x = x^*$ and the C^*-identity we have

$$\|(x^*)^{2^k}(x)^{2^k}\|^{\frac{1}{2^{k+1}}} = (\|x^{2^k}\|^2)^{\frac{1}{2^{k+1}}} = \|x^{2^k}\|^{\frac{1}{2^k}} = \|x\|$$

by the induction hypothesis, which completes the proof. □

If x is self-adjoint, $\|\hat{x}\|_\infty = \|x\|$.

Proof: Fix $x \in A$. $\|\hat{x}\|_\infty := \max_{\ell \in SpecA} |\hat{x}(\ell)|$. This means that $\|\hat{x}\|_\infty := \max_{y\in \mathrm{Ran}(\hat{x})} |y| = \max_{y\in\sigma(x)} |y|$, since $\mathrm{Ran}(\hat{x}) = \sigma(x)$. Now, recognize that $SpecA$ is compact, and so its image is compact in $\mathbb{C}$. Then, by Heine-Borel, its image is closed and bounded in $\mathbb{C}$. In particular, this means that $\max_{y\in\sigma(x)} |y| = \sup_{y\in\sigma(x)} |y| = r(x) = \|x\|$, since x is self-adjoint. □

Now, we will prove that $\hat{}$ is an isometry, meaning $\forall x \in A$, $\|\hat{x}\|_\infty = \|x\|$. This will immediately lead to the injectivity of $\hat{}$.

$\hat{}$ *is an isometry.*

Proof: Fix $x \in A$. Then, $\|\widehat{x^*x}\|_\infty = \|x^*x\|$, since x^*x is self-adjoint. Now, $\|x^*x\| = \|x\|^2$ by the C^*-identity. So we have that $\|\widehat{x^*x}\|_\infty = \|x\|^2$. This means that $\sup_{\ell\in SpecA} |\ell(x^*x)| = \|x\|^2$. But,

$$\sup_{\ell\in SpecA} |\ell(x^*x)| = \sup_{\ell\in SpecA} |\ell(x^*)\ell(x)| = \sup_{\ell\in SpecA} |\ell(x)^*\ell(x)|$$

which again by the C^*-identity (now on $\mathbb{C}$) gives

$$\sup_{\ell \in SpecA} |\ell(x)|^2 = \|x\|^2 \Leftrightarrow \sup_{\ell \in SpecA} |\ell(x)| = \|x\|$$

This means that $\|\hat{x}\|_\infty = \|x\|$. □

$\hat{}: A \to C(SpecA)$ *is injective*

Proof: Consider $\hat{x}, \hat{y} \in C(SpecA) \ni \hat{x} = \hat{y}$. We will prove that $x = y$. $\hat{x} = \hat{y} \Rightarrow \hat{x} - \hat{y} = 0 \Rightarrow \widehat{x - y} = 0$, since $\hat{}$ is linear. Then, $\|\widehat{x - y}\|_\infty = \|x - y\| = 0 \Rightarrow x - y = 0$. So $x = y$. □

It remains only to show that $\hat{}: A \to C(SpecA)$ is surjective. To do this, we will employ the Stone-Weierstrass theorem.

Stone-Weierstrass Theorem*: Let Z be a compact Hausdorff space, and let B be a closed $^*-$subalgebra of continuous functions on Z, which separates points of Z and contains the constant function. Then $B = C(Z)$.*

Now, we will try to prove that $\hat{A} := \{\hat{x} | x \in A\}$, which is the range of $\hat{}$, the Gelfand transform, satisfies the conditions of B above; $SpecA$ will function as Z, the compact Hausdorff space. If we can show that the hypotheses of this theorem are satisfied, then we will have shown that the range of $\hat{}$ is equal to $C(\text{Spec}\, A)$, verifying that $\hat{}$ is surjective. We now verify the hypothesis that $\hat{A}$ seperates the points of Spec A.

$\hat{A}$ *separates the points of* Spec A.

Proof: We wish to verify that $\forall \ell_1, \ell_2 \in SpecA, \exists x \in A$ such that $\ell_1 \neq \ell_2 \Rightarrow \hat{x}(\ell_1) \neq \hat{x}(\ell_2)$. We proceed by contradiction.

Assume that $\exists \ell_1, \ell_2 \in SpecA \ni \ell_1 \neq \ell_2$ and $\forall x \in A, \hat{x}(\ell_1) = \hat{x}(\ell_2)$. This means that $\forall x \in A, \ell_1(x) = \ell_2(x)$. This, of course, means that these functionals are equal, since they have the same domain and are equal over this space, contradicting that $\ell_1 \neq \ell_2$. This proves that $\hat{A}$ separates the points of $SpecA$. □

It remains only to verify that $\hat{A}$ is closed, for then all of the hypotheses are satisfied.

$\hat{A}$ *is closed.*

Proof: We want to show that $\hat{A}$ is closed by showing that it contains all of its limit points. Thus, we consider an arbitrary limit point $\hat{x}$. For $\hat{A}$ to be closed, we must show that $x \in A$. Now, since $\hat{x}$ is a limit point, there is some net

$(\hat{x}_\alpha)_{\alpha \in J} \to \hat{x}$. Then, since $\hat{}$ is injective, we can construct an equivalent net in the domain, X. We must show that this net, $(x_\alpha)_{\alpha \in J} \to x \in A$. Since $\hat{}$ is an isometry, we know that it preserves distances. In particular, if a net converges in one set, it must converge in the other. Thus, $(\hat{x}_\alpha)_{\alpha \in J} \to x \in A$, so then $\hat{A}$ is closed. $\square$

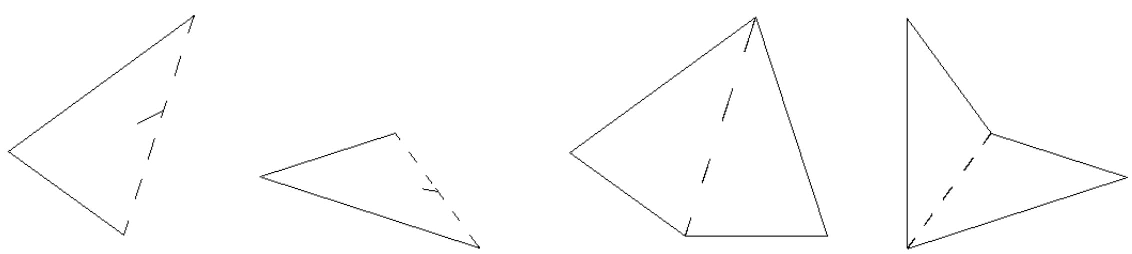
We have shown that $\hat{A} = C(\operatorname{Spec} A)$, which proves that $\hat{} : A \to C(\operatorname{Spec} A)$ is surjective. Thus, $\hat{}$ is a $^*-$homomorphic bijection, and establishes that A is isomorphic to $C(\operatorname{Spec} A)$.

4 Penrose Tilings

We will begin our discussion of Penrose Tilings by introducing what we mean by a tiling of the plane. We will define a two-dimensional tiling, or tesselation, as a partitioning of the plane into a finite set of prototiles. Each prototile is a finite subset of the plane which is homeomorphic to the unit sphere.

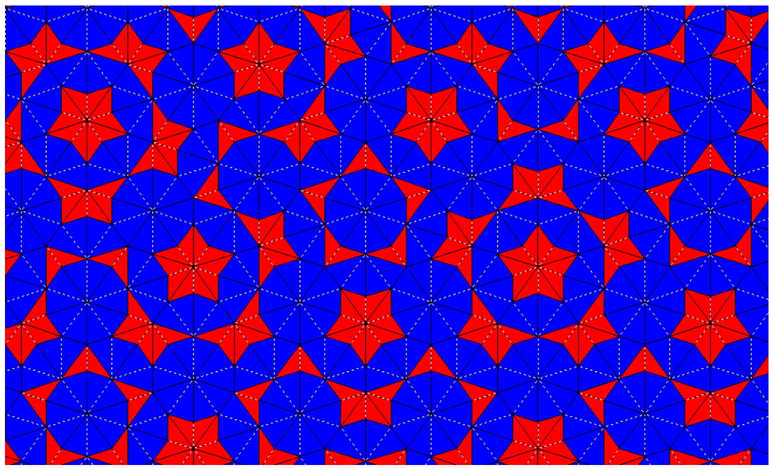
A familiar example of a tiling might be a grid of squares. In this case there is only one prototile. Another property of this simple tiling is translational symmetry. We say that a tiling exhibits translational symmetry if there exists a non-zero vector of translation which maps the tiling identically to itself.

If a tiling does not have vectors of translational symmetry, we say that the tiling is aperiodic. Penrose tilings are perhaps the simplest aperiodic tilings, because they consist of only two prototiles. The prototiles are each isosceles triangles, one with acute and one with obtuse vertex angles, respectively. There is a matching condition as indicated in the figures below.

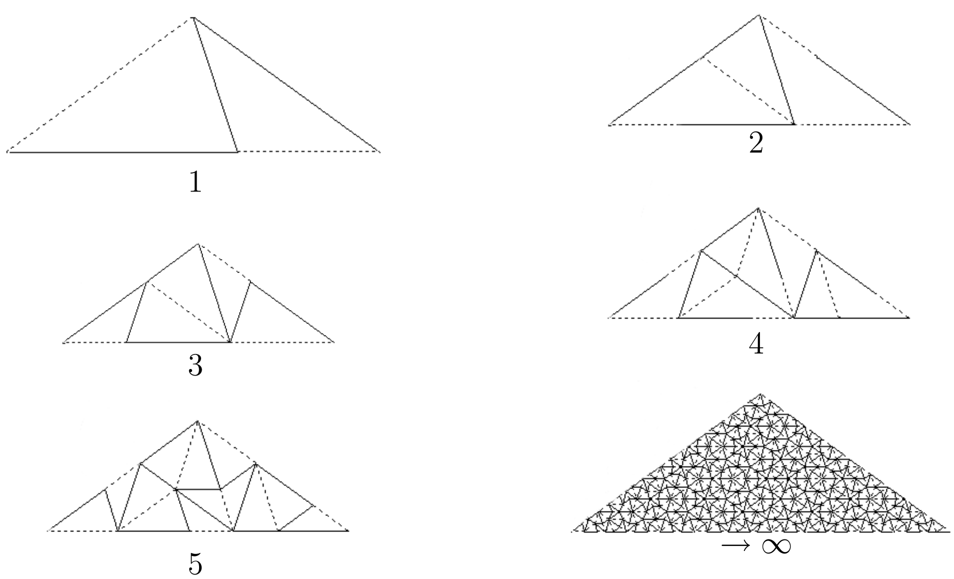


Notice that there are only two different edge lengths in the figure. If we consider the shorter length in the diagram to be the unit, then we find that the other length is $\tau = \frac{1+\sqrt{5}}{2}$, the Golden ratio.

Here is an example of a penrose tiling with acute and obtuse prototiles colored blue and red, respectively.



Constructing a Penrose tiling can be an interesting excercise to do by hand. However, the only way to ensure that any Penrose tiling of the entire plane even exists is to define a decomposition. We begin with some collection of prototiles, and we define an algorithm that partitions these prototiles into new prototiles that are geometrically similar, but reduced in size by some scale factor. This decomposition could be followed by magnification by the scale factor. This process, if it exists, could be repeatedly arbitrarily. We will now define this process for Penrose tilings, proving that arbitrarily large Penrose tilings exist.



We summarize the decomposition illustrated above by noting that after two decomposition steps the acute triangle is decomposed into two acute triangles and an obtuse, while the obtuse triangle is decomposed into an obtuse and an acute. Symbolically, we can write

$$A_{n+1} = 2A_n + O_n \quad O_{n+1} = A_n + O_n$$

where A_i (resp. O_i) is the number of acute (obtuse) prototiles after $2i$ decompositions. Also, note that after each step (instead of every two steps) the size of the respective prototiles changes. At stage 2, we see that the obtuse prototile is now larger than the acute. Indeed, at every stage of decomposition, the ratio of the area of the prototiles is τ, however the larger prototile alternates between the acute and the obtuse. This observation is important, and for this reason we will call the larger prototile in a Penrose tiling Big, and the smaller prototile Small.

We note that the altitudes and base of the acute prototile also have the ratio $\tau : 1$ and the obtuse prototile has the ratio $1 : \tau$. This fact can be proven with elementary geometric methods, and so is ommitted. Now, we will determine how much of the tiling is made from acute prototiles and how much is from obtuse. Let λ_n be the ratio of the number of acute prototiles to the number of obtuse, after n decompositions. Then,

$$\lambda_{n+1} = \frac{A_{n+1}}{O_{n+1}} = \frac{2\frac{A_n}{O_n} + 1}{\frac{A_n}{O_n} + 1} = \frac{2\lambda_n + 1}{\lambda_n + 1}$$

The limit as $n \to \infty$ is τ. Since the ratio of areas of the prototiles is τ, the amount of the plane in the acute partition to the obtuse partition must oscillate between $\tau^2 : 1$ and $1 : 1$.

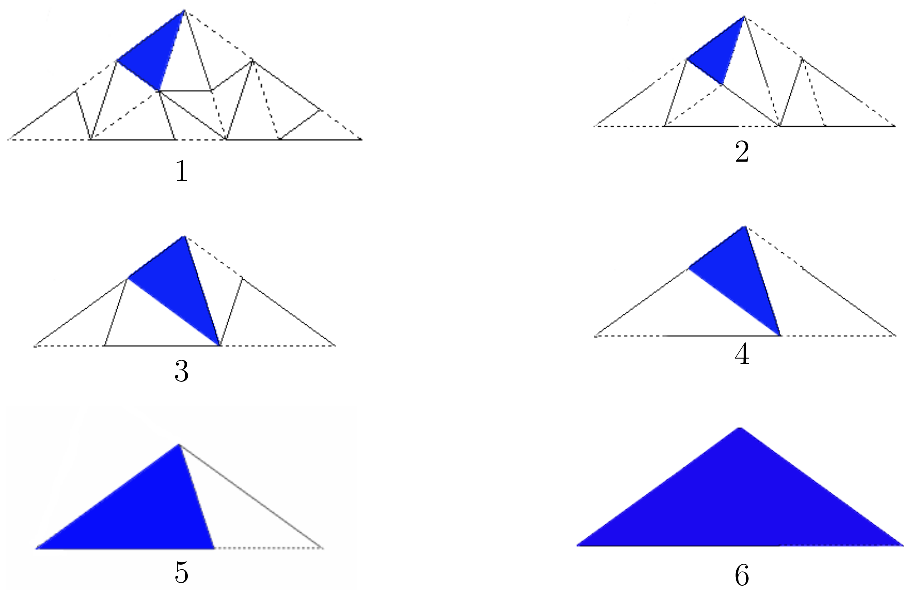
We now have enough information to prove that Penrose tilings are aperiodic. Consider any (infinite) periodic tiling, T, of the plane. Since T is periodic, there exists a finite patch of T (which itself is homeomorphic to the unit ball) such that T is covered by a disjoint union of copies of this finite patch. The ratio of prototiles in the infinite tiling must be exactly equal to the ratio of the prototiles which form the patch. Since this patch is finite, the ratio is necessarily rational. It follows that any tiling of the plane that has an irrational ratio of prototiles can not be periodic.

5 Penrose Universe

In this section we will discuss the set of all Penrose tilings, called the Penrose Universe. Up until this point, we have not ruled out the possibility that there is only one Penrose tiling. To investigate this, we will need to develop a system that can describe a Penrose tiling. First we will have to explore a process called composition.

Composition is essentially the inverse process of decomposition. Instead of separating larger prototiles into smaller ones, we will erase lines to create a tiling with larger prototiles. At each stage, we erase only the lines that would be introduced using the reverse process of composition.

We can describe a Penrose tiling by beginning with some prototile in the tiling, and then composing the tiling one stage. After each composition we note if the new prototile functions as a Big or a Small (B or S) in the current tiling.



We can see in the particular example above, the starting prototile begins as a Big (B). The sequence describing this tiling is then B-S-B-S-B-B.

Complete Penrose tilings can be described in terms of these infinite binary sequences. There is a restriction, however, from all binary sequences, because we notice that in no tiling can a Short prototile be composed into a Short prototile. Therefore, there is a grammar rule in the sequence, where an S is always followed by a B.

Further, we recognize that the choice of initial prototile is arbitrary. For this reason, we must say that two tilings are equivalent if and only if the infinite tails of their binary sequences are equivalent after some finite number of digits in the sequence. We will denote this equivalence relation as $\sim$.

Now, any Penrose tiling can be constructed directly from a sequence satisfying the grammar rules. Thus, if we can produce two sequences which satisfy the grammar rules and do not ever agree, then we will have shown the existence of more than one Penrose tiling. The alternating sequence BSBSBS... and the sequence BBSBBS... are such examples. In fact, it is easy to see now that there are an uncountably infinite number of unique Penrose tilings.

6 Topological Approach

We want to induce a metric on this space of Penrose Tilings. Let X denote the space of infinite binary sequences that satisfy the grammar rule. For this purpose, let B be represented by 1, and S be represented by 0. For $y, z \in X$, let

$$d(y,z) = \sum_{n=1}^{\infty} 2^{-n} |y_n - z_n|$$

where y_i and z_i are the i^{th} digits in the y, z binary sequence, respectively. We assert that $d : X \to \mathbb{R}$ is a metric, though we only verify the triangle inequality.

$\forall x, y, z \in X,\ d(x,y) + d(y,z) \geq d(x,z)$.
Proof: Consider $x, y, z \in X$. We want to show that

$$\sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| + \sum_{n=1}^{\infty} 2^{-n}|y_n - z_n| \geq \sum_{n=1}^{\infty} 2^{-n}|x_n - z_n|$$

Now, using the triangle inequality on $\mathbb{R}$, we know that for all n of the summation, $|x_n - y_n| + |y_n - z_n| \geq |x_n - z_n|$. Our result follows immediately. Similarly, the other two conditions we did not verify are inherited directly from the absolute value metric on $\mathbb{R}$. □

We will now investigate this metric space, (X, d). Remember that we have this space partitioned into equivalence classes based on the equivalence relation stated earlier: if two binary sequences agree completely after some finite number of digits, then they are equivalent.

Consider two distinct Penrose tilings, represented by their sequences $x, y \in X$. Since x, y are distinct, we know that $d(x,y) = r$ for some positive $r \in \mathbb{R}$. We can construct another sequence, x_1, that is equivalent to x and such that $d(x,y) < r$. The construction is the following: find the first digit n where $|x_n - y_n| \neq 0$. Since x and y satisfy the grammar rule, this will not be a digit that is forced (i.e. it will not be a 1 following a 0). Thus, we can let $x_n = y_n$. Clearly, $d(x_1, y) = r - 2^{-n} < r$. However, we need not stop with x_1. Repeating this process arbitrarily, we can see that there exists a Penrose tiling, represented by x_k, which can be made arbitrarily "close to" y under the metric d.

We conclude that every equivalence class is dense in X, since every sequence in X can be made arbitrarily close to a sequence in a given class. This means that if we look at the space $X/\sim$, the set of equivalence classes, we see that it is topologically very strange: the closure of any element in $X/\sim$ is the whole space.

Suppose that we try and construct an abelian algebra related to this space, to try and understand the set algebraically. This algebra, A, would be $\{f \in C(X) \ni f(x) = f(y) \Leftrightarrow x \sim y\}$. However, this would imply that $A \cong \mathbb{C}$, since a continuous function taking the same value on a dense set is constant. So this picture suggests that X is a one-point space; however, we know this set is uncountable.

Thus, neither the topology induced from the metric, nor the abelian algebra associated to the space reveals anything about the structure of the Penrose universe. We will see in the next section that associating a non-abelian algebra to this space is the appropriate method for further study.

7 Algebraic Approach

In this section, we will associate a non-commutative C^*-algebra to the space of Penrose tilings. Then, we will use the methods of algebraic K-theory to classify the associated algebra.

The set of binary sequences that obey the grammar rule, X, has two structures: a metric which gives rise to a topology and an equivalence relation, which partitions the set. We have shown that these structures are not intuitively "consistent," and in general it can be difficult to associate an algebra to such a space. However, we can show that $X/\sim$ is a projective limit of finite spaces, given the decomposition algorithm outlined in section 4. So we will exploit a standard method for associating a matrix algebra with a finite set and equivalence relation.

Now, we will explain what we mean by the projective limit of finite spaces.

$$(X_1, \sim_1) \longleftarrow (X_2, \sim_2) \longleftarrow (X_3, \sim_3) \longleftarrow \cdots \tag{1}$$

Each X_i is the set of all finite binary sequences that satsify the grammer rule, and contain i digits, and each $\sim^i$ is the equivalence relation where the sequences are exactly equivalent after i digits. The projection from $(X_{i+1}, \sim_{i+1}) \longrightarrow (X_i, \sim_i$) is the obvious projection: delete the last digit in each sequence of X_{i+1}. We also note that each space has only two equivalence classes, since the last digit of any sequence in X_i is either a B or an S, which partitions the set.

At each stage we will have two partitions, each corresponding to the last digit of the sequence being 0 or 1. Suppose we are given a finite set, $X_n = \{x_1, \dots x_i, x_{i+1}, x_n\}$ with 2 equivalence classes, and such that $x_j \in$ Partition 1 for $j \leq i$, and else $x_j \in$Partition 2. The, the associated matrix algebra is

$$C^*(X_n, \sim) = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1i} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ii} & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{(i+1)(i+1)} & \dots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{n(i+1)} & \dots & a_{nn} \end{pmatrix} | a_{ij} \in \mathbb{C} \right\}$$

Clearly, this matrix algebra is the direct sum of two other matrix algebras, the rank of each being equal to the number of elements in that partition. Here we find that the rank of these sub-algebras will be the Fibonacci numbers: each sequence in X_i that ends in S must be followed by a B in X_{i+1} because of the grammer rule; however, each series in X_i ending in B can be followed by a B or S. Indeed, since each finite space contains all possible sequences, we can determine the size of each partition in X_i in terms of i:

	X_1	X_2	X_3	X_4	$\cdots$
$B:$	1	2	3	5	
$S:$	1	1	2	3	

Now, we would like to define the C^*-algebra associated to the complete space $X/\sim$. Substituting these matrix algebras for the corresponding finite spaces, we see that diagram (1) translates as

$$C^*(X_1, \sim_1) \longrightarrow C^*(X_2, \sim_2) \longrightarrow C^*(X_3, \sim_3) \longrightarrow \cdots$$

The rigorous details of this translation are omitted, as they are not appropriate for the intended audience. However, we summarize in saying that the projections in the former diagram are translated as injective algebraic homomorphisms. As matrix algebras, we see these (injective, homomorphic) inclusions as

$$\begin{bmatrix} |f_{i+1}| & 0 \\ 0 & |f_i| \end{bmatrix} \longrightarrow \begin{bmatrix} \left| \begin{matrix} f_{i+1} & * \\ * & f_i \end{matrix} \right| & 0 \\ 0 & |f_{i+1}| \end{bmatrix}$$

where the the bars, $||$, indicate sub-algebras.

The non-commutative C^*-algebra associated with $X/\sim$, the space of Penrose tilings, is the injective limit of this sequence of algebras. Now, we would like to study this algebra using the methods of algebraic K-theory.

8 K-Theory

The basic concept of K-theory is invariance; the basic tool of K-theory is the $K_0(X)$ abelian group. Formally, given a unital C^*-algebra, A, we can construct an abelian semi-group $K_0'(A) = (P(A)/\sim_{vn}, \oplus)$. $P(A)$ is the set of all projections, $\{x \in M_\infty(A) \ni x = x^2 = x^*\}$, while $M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$. The equivalence relation, $\sim_{vn}$ is the von-Neumann equivalence, defined for $p, q \in P(A)$, where $p \in M_i(A)$ and $q \in M_j(A)$: $p \sim q$ iff $\exists r : M_i \to M_j(A) \in M_\infty(A) \ni p = uu^*$ and $q = u^*u$. Finally, $\oplus : (P(A)/\sim_{vn})^2 \to P(A)/\sim_{vn}$ is defined such that

$$[p] \oplus [q] = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

Now, we use the Grothendieck construction to create an abelian group from this semi-group. The first part of this construction is to impose an equivalence relation on $K_0'(A)$ that will insure that this semi-group has the cancellation property, which is a necessary condition for the second step. We define a new abelian semi-group, $K_0^+(A) = K_0'(A)/\sim_c$. For $p, q \in K_0'(A), p \sim_c q$ iff $\exists r \in K_0'(A) \ni p \oplus r = q \oplus r$.

Now, we will construct an abelian group, $K_0(A)$ from $K_0^+(A)$. This second part of the Grothendieck construction is defined by $K_0(A) = K_0^+(A)^2/\sim_g$, where $(p, q) \sim_g (r, s)$ iff $p \oplus s = q \oplus r$. For background, this second part of the construction is how one generates the integers as an abelian group $(\mathbb{Z}, +)$ from the abelian cancellable semi-group $(\mathbb{N}, +)$. This latter example is pictured below, for intuitive understanding.

−3	−2	−1	0	1	2	3
(0,3)	(0,2) (1,3)	(0,1) (1,2) (2,3)	(0,0) (1,1) (2,2) (3,3)	(1,0) (2,1) (3,2)	(2,0) (3,1)	(3,0)

$K_0(C^*(X/\sim))$

We would like to calculate the K_0 group for the C^*-algebra associated with $X/\sim$. Though one could construct this from the definitions provided in the previous section, we will note and utilize some useful well-known results in basic K-theory to simplify the discussion.

1) $K_0(M_i(\mathbb{C}) \oplus M_j(\mathbb{C})) = (\mathbb{Z}^2, +)$
2) $K_0(\lim_{\rightarrow}(C_i)) = \lim_{\rightarrow}(K_0(C_i))$

This second result, the fact that the K_0 groups "commute" with the limit is particularly useful, however we have to be sure that the limit of a sequence of abelian groups is well defined. In this case, it will not be a problem.

$$K_0(C*(X/\sim)) = K_0(\lim_{\rightarrow}(M_{f_i} \oplus M_{f_{i+1}})) = \lim_{\rightarrow i \in \mathbb{N}}(K_0(M_{f_i} \oplus M_{f_{i+1}})) = \lim_{\rightarrow i \in \mathbb{N}}(\mathbb{Z}^2, +)$$

This invariance is obviously not a complete invariance; any sequence other than the Fibonnaci sequence used to define the ranks of the respective subalgebras would also produce this K_0 group. Algebraists have developed further constructions, like this K_0 construction, to provide additional structure to differentiate (the isomorphism class of) this algebra from other algebras.

The first piece of additional structure is an order relation on this group which reflects the K_0 construction. The method is as follows: using the abelian cancellable semi-group, $K_0^+(C^*(X/\sim))$, we can say that for $p, q \in K_0(C^*(X/\sim))$,

$$p \geq q \text{ iff } p - q \in K_0^+(C^*(X/\sim))$$

It will be useful to explicitly determine what this 'positive' subset, $K_0^+ \subset K_0 = \mathbb{Z}^2$, actually contains.

We see that the transition matrix associated to consecutive algebras, i.e. $C^*(X_i/\sim_i), C^*(X_{i+1}, \sim_{i+1})$, is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Thus, we want to know the points of $\mathbb{Z}^2$ which will be 'positive,' i.e. end up in $\mathbb{N}^2$ after a finite applications of the transition matrix. The standard method calls for the eigenvalues and eigenvectors of the transition matrix to help determine a phase portrait. We find that

$$\lambda_1 = \tau \qquad \mathbf{v}_1 = \begin{bmatrix} \tau \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{-1}{\tau} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\tau \end{bmatrix}$$

It follows that $K_0(C^*(X/\sim) = \{(x, y) \in \mathbb{Z}^2 \ni \tau x + y \geq 0\}\ )$

Unfortunately, this ordered group, $(K_0(C^*(X/\sim)), K_0^+(C^*(X/\sim)))$, called the dimension group, is still not a complete invariant. To produce the complete invariant, another set, called the scale is introduced and associated with the algebra. One definition of the scale is $\Gamma(A) = \{p \in K_0^+(C*(X/\sim)) \ni p \leq 1_A\}$. In this case, we find that the scale is $\Gamma = \{(x, y) \in \mathbb{Z}^2 | 0 \leq \tau x + y \leq \tau + 1\}$. With this final algebraic characteristic, we have the complete invariant of $C^*(X/\sim)$, the algebra associated with the space of all Penrose tilings.

We notice the multiple appearance of τ in this algebraic invariant. Given τ's multiple roles in the construction of Penrose tilings, we see that the study of this non-commutative algebra gave insight into the structure of the space of Penrose tilings.

9 Bibliography

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