Asymptotic Analysis

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Euler Approximated 1000!

Leonard Euler (1707-1783) used a rapidly convergING (but divergent) series to determine that 1000! had 2568 digits. He used a what is now called Stirling's formula[1], which he derived by a method using what is now called the Euler-Maclaurin summation formula. In this approximation he only used two terms from the approximating series. Questions: How close can we get to the actual values of a function ($n!$ in this case) using asymptotic expansions like Stirling's formula? How do asymptotic formulas behave at (extremely) large values?

Example: Stirling's Asymptotic Formula

$$n! \approx \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{B_2}{(2)(1)n} + \frac{B_4}{(4)(3)n^3} + \frac{B_6}{(6)(5)n^5} + \cdots + \frac{B_{2k}}{(2k)(2k-1)n^{2k-1} + \cdots}}$$

In this equation, B_k are the Bernoulli numbers.[2] Now, Let for $S(x,m)$ represent the mth term approximation of Stirling's asymptotic formula

$$S(x,m) = \sqrt{2\pi x}(\frac{x}{e})^x e^{\frac{B_2}{(2)(1)x} + \frac{B_4}{(4)(3)x^3} + \frac{B_6}{(6)(5)x^5} + \cdots + \frac{B_{2m}}{(2m)(2m-1)x^{2m-1}}}$$

A simple computer experiment can show the accuracy of Stirling's approximation.[3]How many terms does it take to compute 50! precisely? By "precisely" we mean to say that $S(x,k)$ is a precise approximation to $x!$ if there is only one unique $a \in \mathbb{N}$ such that a is in the inteval $[S(x,k), S(x,k+1)]$. SO, for 50, 26 is a precise approximation because there is only one unique integer in between $S(50,26)$ and $S(50,27)$. Concretely, for $x = 50$ we have the unique natural number

$$a = 50! = 30414093201713378043612608166064768844377641568960512000000000000 \quad (0)$$

In between $S(50,26)$ and $S(50,27)$.

[1] Wrongly attributed James Stirling (1692-1707) and appears in *Miscellanea Analytica* (1730) by Abraham de Moivre (1667-1754) more information can be found at http://www.york.ac.uk/depts/maths/histstat/demoivrenotes.pdf and http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Stirling.html

[2] it may be beneficial to recall that the EVEN, nontrivial, Bernoulli numbers are alternating.

[3] the following figures are taken directly from: Pengelley, David.*Dances between continuous and discrete: Euler's summation formula*, Euler 2k+2 Conference. Department of Mathematical Sciences, New Mexico State University. November 5, 2002.

When does the smallest (unique) k where $x! \in [S(x,k), S(x,k+1)]$ occur? Let's compare the smallest order approximation for 50! and 100!. That is, find the order of the approximation which Stirling's Approximation "traps" 50! and 100! respectively. As seen previously, To approzimate 50! we need to take the 26th and 27th order approximation. Doing the same process for 100! we find the smallest order approximation k for which $S(x,k)$ and $S(x,k+1)]$ "trap" 100! is 74; $100! \in S(x,74), S(x,75)]$. Note that doubling 50 requires *more than double* the number of terms to approximate 50! to the nearest integer. This $x = 2(50) = 100$ does NOT have a smallest order approximation of $2(26) = 52$. It can be shown that the minimal order approximationing number as a function of x is nonlinear. This can be verified by trying to make a line through the points $(1,2)$, $(50,26)$, and $(100,74)$. The point $(1,2)$ is used because it requires a second order approximation to compute 1!.

Futher, It is also known that Stirling's approximation will improve (the error will decrease) for roughly the first πx terms of the approximation.[4] It now follows that since the order of k is not linear with x AND Since the approximation impoves for the first πx terms then we should suspect that for some x_0 Stirling's formula cannot find a k where there is a unique integer $x_0! \in [S(x,k),(x,k+1)]$. This result has been confirmed numerically.

Research

The proposed research would include but is not limited to:

- Derivation of the asymptotic Stirling's formula using Euler-Maclaurin summation formula. Derivation of Stirling's formula using the method of steepest decent. Study of asymptotic analysis of Parametric integrals.
- Prove that for there exists a $K \in \mathbb{N}$ such that for all $x \geq K$ there can be no $r \in \mathbb{N}$ where $x!$ is the unique integer in between $S(x,r)$ and $S(x,r+1)]$.
- Applying the techniques to other asymptotic expansions: this would include the application of asymptotic methods of parametric integrals and the application of these methods to the integral representation of Taylor Polynomials.

[4]see above reference