Research Midterm Report for the Fall of 2005
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GENERALIZATIONS OF CONTINUED FRACTIONS

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Abstract. We investigate continued fractions to approximate numbers in the complex plane. We develop an algorithm to approximate complex numbers, then prove some analogous theorems to those that exist for the real case. Also, we analyze the precision as one carries out more steps. Finally we present some conjectures to how the continued fraction of certain classes of numbers behave in the complex plane.

1. Background

Continued fractions have been around for hundreds of years, dating back to Leonardo Fibonacci in his work *Liber abaci* published in 1202 [1]. One way to think of the continued fraction of a real number is to consider it as an extension of the Euclidean algorithm. We are able to express any real number as a sequence of integers

$$x = [a_1, a_2, a_3, a_4, ...] \,,\; a_i \in \mathbb{Z}$$

where

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\vdots}}}}.$$

This is a simple continued fraction representation, that is with 1's in the numerator. For the rest of the paper when we write continued fraction we will mean simple unless otherwise noted.

In our paper we first propose a well known algorithm for computing continued fractions in the reals. We discuss some important theorems and results that hold in the reals. We then explore what changes are needed to have the algorithm function properly for calculating complex continued fractions. We then go on to prove analogous results to the real case for the complex case.

We then propose some future lines of research to be completed. These include some conjectures on how complex continued fractions behave. We also will examine some predictions for how precise a continued fraction approximation is, along with how the rate of precision grows with the number of steps taken. Then we conclude some potential further generalizations that could be made to continued fractions.

First let us explore how to compute a continued fraction in the real case. It is of note how similar this algorithm is to that of the Euclidean algorithm.

2. Algorithm

Input: α and number of steps n
1. $a_1 := \lfloor \alpha \rfloor$
2. $p_1 := a_0$
3. $q_1 := 1$
4. for i from 1 to n {
5. $\quad \alpha := \frac{1}{\alpha - a_i}$
6. $\quad a_{i+1} := \lfloor \alpha \rfloor$
7. $\quad$ if $i = 1$ then
8. $\qquad p_i := a_1 a_2 + 1$
9. $\qquad q_i := a_2$
10. $\quad$ else
11. $\qquad p := a_i a_{i+1} + p_{i-1}$
12. $\qquad q := a_{i+1} q_i + q_{i-1}$
13. }
14. output $\{a_n\}$, $\{p_n\}$, $\{q_n\}$

The sequence $\{a_n\}$ is what is to be expected, that is the sequence that represents the continued fraction. Now $\frac{p_n}{q_n}$ is the approximation of the number α we have calculated to the n^{th} step. What is of note of this approximation is that there is no denominator less than q_n that provides a better approximation.

In fact we have that

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

However, it is much faster to keep track of the p_i and q_i throughout the algorithm than to compute them through matrix representation later.

Now we will consider some important algorithms for the real case.

3. Known Theorems for the Reals

Theorem 3.1. *A simple continued fraction is finite if and only if it represents a rational number.*

Theorem 3.2. *Every irrational number has a unique infinite simple continued fraction representation.*

Theorem 3.3. *A simple complex continued fraction is periodic if and only if it is the solution to a quadratic equation with integer coefficients.*

4. Computing Complex Continued Fractions

Now let us examine our original algorithm for computing continued fractions. We will be specific about what operations we performed. On line $1, 6$ we take the floor function of the number we are given to compute its continued fraction. Then on line $5, 8, 11, 12$ we use addition along with on line 5 using an additive inverse. Also on lines $8, 11, 12$ we use multiplication along with on line 5 taking a multiplicative inverse. All these operations besides the floor function exist in any field.

Now let us explore what it means to find the greatest integer less than a number. First it is necessary to determine what we mean by an integer. In the complex numbers for an integer we mean a Gaussian integer that is a number of the form

$a + bi$ with $a, b \in \mathbb{Z}$ and $i^2 := -1$. Let z be a complex number. Naively we could take the floor to be

$$\lfloor z \rfloor := \lfloor \Re z \rfloor + \lfloor \Im z \rfloor i.$$

However, this is not a good floor since there may exist Guassian integers greater than what we found that are still less than z.

We will present the algorithm we use to find the floor of a complex number, then we will examine it.

Input: z
1. $a := \Re z - \lfloor \Re z \rfloor$
2. $b := \Im z - \lfloor \Im z \rfloor$
3. if $a + b < 1$ then $x := 0$
4. if $a + b >= 1$ and $a >= b$ then $x := 1$
5. if $a + b >= 1$ and $a < b$ then $x := i$
6. output $\lfloor \Re z \rfloor + i \lfloor \Im z \rfloor + x$

This floor gives us the largest Guassian integer that has magnitude less than the input while still remaining within a radius of one of our input. This takes care of finding a Gaussian number that doesn't have one greater than it, yet keeps us close to the number, which is exactly what we want. However, there is one important difference between this and the floor for a real number. The floor function is in fact a function, that is it gives a unique integer as the answer. In contrast the floor for a complex number is a relation and not a function, it is arbitrary that we chose when $a = b$ to add 1 we could have just as easily decided to add i and have gotten what we "wanted". The floor function for the reals returns a greatest integer, the complex returns a maximal Gaussian integer. Now let us examine how this affects some of the analagous theorems.

5. Analogous Theorems

Conjecture 5.1. A simple continued fraction is finite if and only if it represents a rational number in the complex plane, i.e. a number of the form $r + si$ with $r, s \in \mathbb{Q}$.

Conjecture 5.2. Every irrational number has an infinite simple continued fraction representation.

This is no longer necessarily unique in general since the floor in the complex plane is not unique.

Conjecture 5.3. A simple complex continued fraction has a periodic representation if and only if it is the solution to a quadratic equation with Guassian integer coefficients.

We will go on to prove these results in our follow up to this paper.

6. A Small Example

Let us find the the continued fraction representation of $e^{\frac{2}{3}\pi i}$, which will be periodic since it is the root of a degree two polynomial with Guassian integer coefficients.
Set $\alpha := e^{\frac{2}{3}\pi i} = -.5 + (.8660\ldots)i$
$a_1 := \lfloor \alpha \rfloor = -1 + i$
$\alpha := \frac{1}{\alpha - a_1} = 1.8660\ldots + .5i$

$a_2 := \lfloor \alpha \rfloor = 2$
$\alpha := \frac{1}{\alpha - a_2} = -.5 - (1.8660\ldots)i$
$a_3 := \lfloor \alpha \rfloor = -1 - 2i$
$\alpha := \frac{1}{\alpha - a_3} = 1.8660\ldots - .5i$
$a_4 := \lfloor \alpha \rfloor = 2 - i$
$\alpha := \frac{1}{\alpha - a_4} = -.5 - (1.8660\ldots)i$

Since α is now back to being equal to what it once was we know that the rest of the solution will be periodic. Thus

$$e^{\frac{2}{3}\pi i} = -1 + i + \cfrac{1}{2 + \cfrac{1}{-1 - 2i + \cfrac{1}{2 - i + \cfrac{1}{-1 - 2i + \cfrac{1}{2 - i + \cfrac{1}{\vdots}}}}}}$$

7. Future Work

Now that we have found an appropriate function that computes complex continued fractions and have numerical evidence that our conjectures hold we must prove those.

We will study the rate of convergence of our algorithm, and determine some estimates for how precise it is for how many steps are carried out. It is also worth exploring whether there exist another function that converges faster than the one we have already found.

Since we have implemented our algoirthm into maple. We will explore if there any interesting patterns that could be detected for some continued fraction. We already know that numbers that can be written as a degree two polynomial with Gaussian integers have periodic continued fraction representations.

A further step would be to consider other contexts in which the concept of continued fractions can be generalized. One can begin with an integral domain D, an associated quotient field F and an algebraic extension K of F all with an appropriate topology. If D is an integral domain, then an element $d \in D$ is a divisor of an element $c \in D$ if there exists a $b \in D$ such that $d = bc$. An element a is a greatest common divisor of b and c if given any other divisor d of b and c we have that d is a divisor of c. In an integral domain is called a greatest common divisor domain if there exists a greatest common denominator for all pairs of nonzero a and b. The next thing we might desire is a way to express the greatest common factor of a and b as a linear combination of a and b. The integers, for example satisfy this property. Any greatest common divisor domain which satisfies the second property is known as a Bezout domain. From here, we would like to ask the same questions that we asked for continued fractions of complex numbers in terms of Gaussian integers.

References

[1] Strayer, James K. *Elementary Number Theory.* PWS Publishing Company 1994.