

# “An Excursion into Set Theory using a Constructivist Approach”

Midterm Report

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## Abstract

Constructive logic is an alternative to the theory of classical logic that draws a direct relation between an assertion and its existence. Some of the few important differences between a classical proof and a constructionist proof include the avoidance of proof by contradiction and the rejection of the law of the excluded middle. However, given the possibility that constructive logic is an alternative in proof theory, is this approach to proof an effective alternative to understanding important theorems that have already been classically proven?

To motivate an answer, this project shall introduce the basic foundations of Set Theory, a platform upon which we intend to apply constructivist logic. We will then reconstruct important definitions from a constructivist point of view.

## Informal and Formal Notions of Constructive Mathematics

The birth of modern constructive logic is often attributed to the doctoral dissertation of L.E.J Brouwer (1881-1966) titled *Over de Grondslagen der Wiskunde* which was published in 1907. In his paper, Brouwer introduced the idea of *intuitionism*, a fact when applied to mathematics that demands a direct relation between the validity of an abstract mathematical structure to the existence of its construction.

To introduce, informally, important concepts in constructive mathematics, one only needs to begin with fundamental concepts of classical logic that form the basis of many theorems in classical mathematics. For example classically, it is possible to prove that  $P \vee Q$  holds without explicitly showing that  $P$  holds or that  $Q$  holds. However, an intuitionist approach to the disjunction rule would require that it is not enough to assert  $P \vee Q$  without either proving the existence of  $P$  or proving the existence of  $Q$ .

Keeping the rule of disjunction in mind, how would we tackle a situation when  $Q = \neg P$ ? This situation leads to the well known *Law of the Excluded Middle* or LEM that states that for any proposition  $P$  it is true that  $(P \vee \neg P)$ . While at first glance the proposition seems ordinary and uninteresting, one should realize that most often we will not know either  $P$  or  $\neg P$ . A famous example is Fermat's Last Theorem. If we were to consider this open mathematics problem before its solution was introduced in 1994 by Andrew Wiles, could we really assert  $(F \vee \neg F)$  without having proven  $F$  or  $\neg F$ ? It is this idea that led constructivist mathematician L.E.J Brouwer to reject LEM from any form of constructivist proof.

It is important at this stage to emphasize that a constructivist proof often requires a finite routine with a set of rules that will not rely on the idealistic notion of disjunction as implied by LEM and solely be rested on the fundamental notion of existence. Thus the following rules<sup>†1</sup> are an introduction to what possibly one can use to prove an assertion constructively:

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|--------------------------------------|--|
| (a) $\vee$ ( <i>or</i> )             | to prove $P \vee Q$ we must either construct a proof of $P$ or a proof of $Q$                        |
| (b) $\wedge$ ( <i>and</i> )          | to prove $P \wedge Q$ we must prove both existence of $P$ and $Q$                                    |
| (c) $\rightarrow$ ( <i>implies</i> ) | a proof of $P \rightarrow Q$ is an algorithm that converts a proof of $P$ into a proof of $Q$        |
| (d) $\neg$ ( <i>not</i> )            | to prove $\neg P$ we must show that $P$ implies $0=1$  |
| (e) $\exists$ (there exists)         | to prove $\exists x P(x)$ we must construct an object $x$ and prove that $P(x)$ holds                |
| (f) $\forall$ (for all)              | a proof of $\forall x P(x)$ , is an algorithm, applied to any object $x$ , proves that $P(x)$ holds. |

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<sup>†1</sup> – Constructive Interpretation of Logic, Stanford Encyclopedia of Philosophy

## Brief Introduction to the Theory of Sets

In this project, set theory will be introduced in its *naïve form*. *Naïve set theory* is simply the theory of sets that “*are introduced and understood using what is taken to be the self evident concept of sets as collections of objects considered as a whole*”<sup>†2</sup>. On the other hand, *axiomatic set theory* is the theory of sets that are defined and introduced by first defining the *axioms* that define their properties. This form of set theory, often known as ZFC or Zermelo-Fraenkel-Choice Set Theory, shall not be considered during the course of this project.

The concept of a set is fundamental to mathematics and is defined by the group elements that constitute that set. A function  $f$  is defined as a rule  $f: A \rightarrow B$  that associates an element in  $b \equiv f(a)$  of a set  $B$  to each element  $a$  of a set  $A$ . Thus the set of all functions from a set  $A$  to a set  $B$  will be denoted  $F(A,B)$ .

It is important to understand that non-countable sets are of little practical importance, since it would be impossible to validate the existence of such a structure.

**Definition 1** Let  $A$  and  $B$  be sets. Thus,  $A$  is considered to be a subset of  $B$  if and only if every element in  $A$  is an element of set  $B$ . This is denoted in abbreviated form as  $A \subseteq B$ . If on the other hand, the set  $A$  is indeed a subset of  $B$  and there is at least one element in  $B$  that is not contained in  $A$ , then  $A$  is called a proper subset of  $B$ . Sets  $A$  and  $B$  are said to be equal if and only if  $A \subseteq B$  and  $B \subseteq A$ .

**Definition 2** Let  $A$  and  $B$  be sets. Thus, the union of sets  $A$  and  $B$  is the set of elements that are not shared by both sets and is often denoted  $A \cup B$ . The intersection of sets  $A$  and  $B$ , denoted  $A \cap B$ , is the set of all elements that are shared by both sets  $A$  and  $B$ .

**Definition 3** Suppose there exists a non empty set  $A$  containing for each  $\lambda$  a set  $A_\lambda$ , then 
$$\zeta = \{ A_\lambda : \lambda \in A \}$$
 is called an *indexed family* of sets with  $\Lambda$  as the index set. Thus, the union of all sets contained in  $\zeta$  is given by

$$\bigcup \{ A_\lambda : \lambda \in A \} = \{ x : x \in A_\lambda \text{ for some } \lambda \in A \}$$

Likewise, the intersection of all sets contained in  $\zeta$  is given by

$$\bigcap \{ A_\lambda : \lambda \in \Lambda \} = \{ x : x \in A_\lambda \text{ for all } \lambda \in A \}$$

**Definition 4** Let  $A$  and  $B$  be sets. Then the Cartesian product (also called cross product) of  $A$  and  $B$ , denoted  $A \times B$  is the set of all ordered pairs  $(a,b)$  such that  $a \in A$  and  $b \in B$ . This can be written as

$$A \times B = \{ (a,b) : a \in A \text{ and } b \in B \}$$

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<sup>†2</sup> –Set Theory, Wikipedia, <http://en.wikipedia.org>

**Definition 5** The complement of set  $B$  in  $A$ , denoted  $A \setminus B$ , is the set of all elements in  $A$  and not in  $B$ . If the intersection of sets  $A$  and  $B$  constitute the empty set  $\emptyset$  i.e.  $A \cap B = \emptyset$ , then both  $A$  and  $B$  are disjoint

**Definition 6** Let  $S \subseteq \mathbf{R}$ . If by definition  $S \subseteq S$ , then  $S$  is called a closed set. If,  $S \subseteq \mathbf{R} \setminus S$ , then  $S$  is called an open set. The following are also true:  
 (a) the union of any collection of open sets is an open set  
 (b) the intersection of any finite collection of open sets is an open set

### Constructive Set Theory

To introduce important ideas in constructive set theory, let us consider the notion of complemented sets. In particular we shall introduce, by example, exceptions which are based on the idea of complemented sets.

The idea of complementation is important since most of its proofs can be defined by proof of contradiction which is not allowed from constructivist point of view. Thus, it is necessary to define complementation using the notion of complemented sets, and using its properties in such a way as to avoid any usage of proof by contradiction. One such way is use the notion of inequality with complemented sets.

**Definition 7<sup>†3</sup>** Suppose  $X$  is a set and  $F$  a non-empty set of functions from  $X$  to  $\square$ . A complemented set in  $X$  consists of an ordered pair  $(A, B)$  of subsets of  $X$  such that for each  $x$  in  $A$  and each  $y$  in  $B$  there is exists  $f$  in  $F$  with inequality  $f(x) \neq f(y)$ . Thus a union of such a pair denoted  $\bigcup_{t \in T} (A_t, B_t)$  is defined by

$$A \equiv \bigcup_{t \in T} A_t \quad B \equiv \bigcap_{t \in T} B_t$$

While the intersection denoted  $\bigcap_{t \in T} (A_t, B_t)$  is defined by

$$A \equiv \bigcap_{t \in T} A_t \quad B \equiv \bigcup_{t \in T} B_t$$

The following rules are valid for complemented sets:

$$\neg(\neg A) = A \quad (a)$$

$$\neg \bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} (\neg A_n) \quad (b)$$

$$\neg \bigcap_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (\neg A_n) \quad (c)$$

Thus given the notion of complemented sets, the following introduces exceptions to some laws involving complemented sets and what it means to constructively decide an assertion of a mathematical statement.

The infinite distributive law defined in Bishop & Bridges 1985<sup>†4</sup>, is shown on the following page:

$$U \cup \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (U \cup U_n) \quad (1)$$

is not valid for complemented sets  $U$  and  $U_n$ . To show this, one can consider the set  $U$  defined as:

$$U \equiv \bigcup_{n=1}^{\infty} \neg U_k$$

Now, let the set  $\Gamma \equiv \square$  and a sequence  $(n_k)_{k=1}^{\infty}$  denote a sequence of integers constituting either a 1 or a 0 for each  $k$ . We also do not know whether  $n_k$  is a 0 for a particular  $k$ . Now we define the  $k$ th complemented set  $U_k$  with an ordered pair  $U_k = (\square, \emptyset)$  by the following:

$$\begin{aligned} U_k &= (\square, \emptyset) \quad \text{if } n_k = 1 \\ U_k &= (\emptyset, \square) \quad \text{if } n_k = 0 \end{aligned}$$

Now the union of sets  $U$  and  $U_k$  gives rise to the ordered pair  $(\square, \emptyset)$  as shown below:

Using (1), we have from LHS:

$$\left[ \left\{ \bigcup_{n=1}^{\infty} \neg U_k \right\} \cup \left\{ \bigcap_{n=1}^{\infty} U_k \right\} \right] = \bigcap_{n=1}^{\infty} \{U \cup U_k\} = (\square, \emptyset)$$

Since  $U \cup U_k = (\square, \emptyset)$  for each  $k$ . Now,

$$\begin{aligned} (\square, \emptyset) &= \left[ \left\{ \bigcup_{n=1}^{\infty} \neg U_k \right\} \cup \left\{ \bigcap_{n=1}^{\infty} U_k \right\} \right] \\ &= \left[ \left\{ \bigcup_{n=1}^{\infty} \neg U_k \right\} \cup \left\{ \neg \bigcup_{n=1}^{\infty} (\neg U_k) \right\} \right] \quad \text{using rule (c)} \\ &= \left[ \left\{ \bigcup_{n=1}^{\infty} \neg U_k \right\} \cup \left\{ \bigcup_{n=1}^{\infty} (U_k) \right\} \right] \quad \text{using rule (a)} \end{aligned}$$

As one can see, this is analogous to the LEM which is not allowed as part of a constructive proof. Thus it is not possible to find an element in  $\square$  belonging to

$U \cup \bigcap_{n=1}^{\infty} U_n$ . Thus, the relation defined in (1) is not valid.

It must be mentioned here that the reason we demand a constructive basis is simply to avoid usage of negation which is often used in operations involving complemented sets. And as a result, while most laws w.r.t to complemented sets are derived from classical laws, there are often the odd one or two that need to be redefined from a constructivist point of view.

Last but not least, a constructive substitute<sup>†5</sup> to the infinite distributive law (1), is defined below as follows:

$$(U \cup -U) \cap \left( U \cup \bigcup_{n=1}^{\infty} U_n \right) = (U \cup -U) \cap \bigcup_{n=1}^{\infty} (U \cup U_n)$$

### Project Outline and Intended Outcome

Having constructed a platform based on fundamental ideas of set theory, we shall now embark on an excursion that will attempt to reconstruct important theorems/definitions using constructive set theory.

Of interest, and a possible research topic as a motivation to constructive mathematics, I intend to explore a possible constructive proof to the Burali-Forti Paradox which is stated below:

The Burali-Forti Paradox shows that by “naively constructing”<sup>†6</sup> a set of all ordinal numbers, one arrives at a contradiction with respect to its construction. The reason behind this is that by constructing a set of ordinal numbers  $\zeta$ , this set automatically qualifies as having a successor  $(\zeta + 1)$  which is strictly greater than  $\zeta$ , given by the following relation:

$$\zeta < (\zeta + 1)$$

However, a contradiction arises that since  $\zeta$  is the set of all ordinal numbers then  $(\zeta + 1)$  being itself an ordinal number, there by belongs to the set  $\zeta$ . If this is so, then the following relation must also hold:

$$\zeta < (\zeta + 1) \leq \zeta$$

And thus, we have a contradiction.

The aim for the rest of the semester is to explore possible ways to circumnavigate this paradox constructively if at all this paradox exists from a constructive view point and this remains to be seen.

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<sup>†5</sup> –Equation (2.3), *Set Theory*, Constructive Analysis, Errett Bishop & Douglas Bridges, 1985

†6 –Equation (2.3), *Set Theory*, Constructive Analysis, Errett Bishop & Douglas Bridges, 1985