p-adic Continued Fractions

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Abstract

Simple continued fractions in $\mathbb{R}$ have a single definition and algorithms for calculating them are well known. There also exists a well known result which states that $\sqrt{m}$, $m \in \mathbb{Q}$ and $m \geq 0$, always has a periodic continued fraction representation [4]. In $\mathbb{Q}_p$, the field of p-adics, however, no single algorithm exists which always produces a periodic continued fraction for $\sqrt{m}$, and no result is available to guarantee the existence of one. In Jerzy Browkin's 1978 and 2000 papers on p-adic continued fractions, several algorithms for computing continued fraction approximations to p-adic square roots are given. The intention of this paper is to present some interesting results concerning sufficient and necessary conditions for the periodicity of a p-adic continued fraction as well as to provide an introduction to p-adic numbers and the definitions associated with them.

1 Introduction

In $\mathbb{R}$, there are many useful results pertaining to continued fractions: every irrational number has a unique simple continued fraction representation, and all numbers of the form $\sqrt{m}$, $m \geq 0$ have periodic continued fraction representations [4]. In another field, $\mathbb{Q}_p$, we would like to have similarly nice results. However, the situation is not as simple: there are multiple and non-equivalent definitions of continued fractions, and the standard algorithm for finding continued fractions in $\mathbb{R}$ relies on a function which has a level of ambiguity when imposed on $\mathbb{Q}_p$. In this paper, two algorithms from [1, 2] for finding continued fractions in $\mathbb{Q}_p$ are presented. These algorithms have led to an interesting observation concerning the periodicity and periodicity prediction for continued fractions in $\mathbb{Q}_p$ which is discussed in section 5.1.

2 Definitions

Definition 1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence. Then $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy Sequence* if

for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$, $|x_m - x_n| < \epsilon$.

Note that a sequence in $\mathbb{R}$ converges in $\mathbb{R}$ if and only if it is a *Cauchy Sequence* and that an infinite series, $\sum_{i=0}^{\infty} x_i$ with $x_i \in \mathbb{R}$, converges in $\mathbb{R}$ if and only if the sequence defined by its partial sums is a *Cauchy Sequence.*

Definition 2. Let K be a field. A *valuation* on K is a function $|\cdot| : K \longrightarrow \mathbb{R}$ with the following properties:

1. $|a| \geq 0$ for all $a \in K$, and $|x| = 0$ if and only if $x = 0$.
2. $|a \cdot b| = |a| \cdot |b|$ for all $a, b \in K$.
3. $|a + b| \leq |a| + |b|$ for all $a, b \in K$.

Specifically, if we take $K = \mathbb{Q}$, then $a \in \mathbb{Q}$ can be written with prime p as

$$a = \frac{a'}{b} p^n \text{ with } a', b \in \mathbb{Z},$$

with a', b not divisible by p. The *valuation* $|\cdot|_p$ is then given by:

$$|a|_p := \begin{cases} \frac{1}{p^n} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

This *valuation* is called the p-adic *valuation.*

Definition 3. A metric space is a space K with a function $d : K \times K \longrightarrow \mathbb{R}$ such that:

1. $d(x, y) \geq 0$, for all $x, y \in K$.
2. $d(x, y) = d(y, x)$, for all $x, y \in K$
3. $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in K$.

The function d is called the *metric* on K. In particular, $d(x, y) = |x - y|_p$ is a *metric* induced by the valuation $|\cdot|_p$.

Definition 4. Let K be a metric space with metric d. Let $\bar{K}$ be the set of all Cauchy sequences in K. We then define the equivalence relation between Cauchy Sequences, $\sim$: if $\{x_n\}$ and $\{y_n\}$ are Cauchy Sequences, then

$$\{x_n\} \sim \{y_n\} \text{ if } \{x_0, y_0, x_1, y_1, \ldots, x_n, y_n, \ldots\} \text{ is also a Cauchy sequence.}$$

Let $\widehat{K}$ be the set of all equivalence classes in $\bar{K}$. The metric d then extends like so:

$$d(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n).$$

The fact that $\lim_{n\to\infty} d(x_n, y_n)$ is defined is by virtue of the definition of the equivalence operator $\sim$, and is discussed at length in [5]. $\widehat{K}$ is called the *completion* of K. It is important to note that $\widehat{K}$ contains K in the sense that the subset of $\widehat{K}$ consisting of constant sequences is isomorphic to K. In addition to this, $\widehat{K}$ also contains all of the limit points of the Cauchy Sequences in K.

As an example, the *completetion* of the field of rationals, $\mathbb{Q}$, with respect to the metric induced by the standard absolute value, $|\cdot|$, is $\mathbb{R}$.

Definition 5. Given a prime p, the *p-adic integers*, $\mathbb{Z}_p$, are obtained by taking the completion of $\mathbb{Z}$ with respect to the metric induced by the valuation $|\cdot|_p$.

Definition 6. Similar to the definition of p-adic integers, the field of *p-adic rationals*, $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ with respect to the metric induced by the valuation $|\cdot|_p$. This is equivalent to the fraction field of the p-adic integers, $\mathbb{Z}_p$. Note that, as a result of the definition of completion, all $\alpha \in \mathbb{Q}_p$ can be written as

$$\alpha = \sum_{i=-r}^{\infty} a_i \cdot p^i, \ a_i \in \{0, 1, 2, \ldots, p-1\}, \tag{1}$$

or equivalently as

$$\alpha = \sum_{i=-r}^{\infty} a_i \cdot p^i, \ a_i \in \{0, \pm 1, \pm 2, \ldots, \pm \frac{p-1}{2}\}, \text{ for odd prime } p.$$

By writing elements of $\mathbb{Q}_p$ this way, arithmetic on them is simply performed modulo p.

Definition 7. A *simple real continued fraction* is a fraction of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

with $a_i \in \mathbb{N}$, $i \neq 0$, and $a_0 \in \mathbb{Z}$. This *simple real continued fraction* can be equivalently represented as

$[a_0, a_1, \ldots]$ for the infinite case and as $[a_0, a_1, \ldots, a_n]$ for the finite case.

Definition 8. Browkin presents two non-equivalent definitions of a *p-adic continued fraction* in [2]. However, only one of these definitions is used in this paper: A *p-adic continued fraction* is a fraction of the form:

$$\alpha = b_0 + \cfrac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \ldots}}}, \text{ with } b_i \in \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right).$$

As in the definition of a real continued fraction, a *p-adic continued fraction* can be equivalently represented as

$[b_0, b_1, \ldots]$ for the infinite case and as $[b_0, b_1, \ldots, b_n]$ for the finite case.

Hensel's Lemma. *Let K be a complete field with respect to the valuation $|\cdot|$ and let R be the ring of integers in K. Let $f(x)$ be a polynomial with coefficients from R such that*

$$\textit{there exists some } \alpha_0 \in R \textit{ such that } |f(\alpha_0)| \leq |f'(\alpha_0)^2|,$$

where $f'(x)$ is the formal derivative of $f(x)$. Then there exists a unique root of $f(x)$, $\alpha \in K$. Furthermore, the sequence defined by

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)}$$

converges to α.

The extreme similarity between Hensel's lemma and Newton's method should be noted.

3 Methods for Computing p-adic Continued Fractions

The standard algorithm for computing a continued fraction from a real number relies fundamentally upon the floor function. The floor function is a function such that, when passed a real number, returns the greatest integer less than or equal to that number. The problem with extending the standard algorithm in $\mathbb{R}$ to $\mathbb{Q}_p$ is that it is not clear which part of a p-adic number is "decimal" and which part is "integer". For this reason, it is necessary to define a floor function from scratch. However, due to the ambiguities inherent in this, no clear choice for a floor function in $\mathbb{Q}_p$ exists, although there are several reasonable candidates. The difference between the two algorithms presented below and the various algorithms presented in [2] are completely due to this ambiguity. For $\alpha \in \mathbb{Q}_p$, with p odd and α given by

$$\alpha = \sum_{i=-r}^{\infty} a_i \cdot p^i,\ a_i \in \{0, \pm 1, \ldots, \pm \frac{p-1}{2}\}, \tag{2}$$

define the map

$$s : \mathbb{Q}_p \longrightarrow \mathbb{Q} \text{ with } s(\alpha) = \sum_{i=-r}^{0} a_i \cdot p^i,$$

and a_i as in (2). This function will be called the p-adic floor function. Note that this function considers the "integer" part of α to be the part of the summation in (2) with non-positive indicies. This is natural since this portion of a p-adic number is finite and is larger, with respect to the p-adic valuation, than the "decimal" portion of the number.

The first algorithm presented procedes analagously to the standard algorithm for computing a real continued fraction.

3.1 Algorithm I

Given $\alpha \in \mathbb{Q}_p$, we inductively define sequences $\{a_n\}$ and $\{b_n\}$ as follows:

```
[Step 1] $i = 0$.  Let $a_0 = \alpha$ and $b_0 = s(\alpha)$.
[Step 2] If $a_i = b_i$ then $a_{i+1}$ and $b_{i+1}$ are undefined.  If
    this is the case, quit the algorithm.
[Step 3] $i = i + 1$.  Let $a_i = (a_{i-1} - b_{i-1})^{-1}$ and $b_i = s(a_i)$.  Go
    to Step 2.
```

The resulting sequence $\{b_n\}$, $n \in \mathbb{N}$, is defined to be the p-adic continued fraction approximation of α.

3.2 Algorithm II

For $\alpha \in \mathbb{Q}_p$, define

$$s_1(\alpha) = s(\alpha) = \sum_{i=-r}^{0} a_i \cdot p^i$$

with a_i as in (2). As in Algorithm I, define

$$s_2(\alpha) = \sum_{i=-r}^{-1} a_i \cdot p^i \text{ and } s_2'(\alpha) = s_2(\alpha) - sign(s_2(\alpha)).$$

Lastly, let $s_1'' = s_1$ and

$$s_2''(\alpha) = \begin{cases} s_2(\alpha) & v(\alpha - s_2(\alpha)) = 0 \\ s_2'(\alpha) & \text{otherwise} \end{cases}$$

Where v is the valuation with respect to the prime p.

With these new definitions in hand, we procede to define Algorithm II:

> Given $\alpha \in \mathbb{Q}_p$, we inductively define sequences $\{a_n\}$ and $\{b_n\}$ as follows:

```
[Step 1] $i = 0$ Let $a_0 = \alpha$ and $b_0 = s_1''(a_0)$.
[Step 2] If $a_i = b_i$, then $a_i$ and $b_i$ are undefined.  If this
    is the case, quit the algorithm.
[Step 3] $i = i + 1$.  Let $a_i = (a_{i-1} - b_{i-1})^{-1}$ and $b_i = s_2''(a_i)$.
[Step 4] If $a_i = b_i$, then $a_i$ and $b_i$ are undefined.  If this
    is the case, quit the algorithm.
[Step 5] $i = i + 1$.  Let $a_i = (a_{i-1} - b_{i-1})^{-1}$ and $b_i = s_1''(a_i)$.  Go
    to Step 2.
```

> Continuing in this manner, using s_1'' for even i and s_2'' for odd i, we obtain a sequence $\{b_n\}_{n \in \mathbb{N}}$. This sequence is defined to be the p-adic continued fraction approximation of α.

3.3 Computational Considerations

In $\mathbb{Q}_p$, there exist many elements which we would like to, for computational matters, consider as infinite series. Computers, on the other hand, are by their nature finite and thus preclude using truly infinite series. Thus, we must decide upon a level of precision to use. For this paper, all p-adic numbers were computed out to 5000 decimal places. That is, for $\alpha \in \mathbb{Q}_p$

$$\alpha \approx \sum_{i=-r}^{5000} a_i \cdot p^i, \; a_i \in \{0, 1, 2, \ldots, p-1\}$$

was used in place of

$$\alpha = \sum_{i=-r}^{\infty} a_i \cdot p^i,\ a_i \in \{0, 1, 2, \ldots, p-1\}.$$

Clearly, this has the possibility of generating errors. As a consequence of this, all results given in this paper are observational.

4 Additional Algorithms

4.1 Evaluating a Continued Fraction

When finding continued fraction representations of numbers, it is useful to have a way to verfiy that the continued fraction and the number it represents are equal. Thus, a standard algorithm for evaluating continued fractions is introduced [2, 4]:

if the continued fraction $[b_0, b_1, \ldots, b_n]$ is expressed as $A_n \cdot (B_n)^{-1} \in \mathbb{Q}_p$

with

$$\begin{array}{lll} A_0 = b_0, & A_1 = b_0 \cdot b_1 + 1, & \text{and } A_n = b_n \cdot A_{n-1} + A_{n-2}, \text{ for } n \geq 2, \\ B_0 = 1, & B_1 = b_1, & \text{and } B_n = b_n \cdot B_{n-1} + B_{n-2}, \text{ for } n \geq 2. \end{array} \tag{3}$$

Then $A_n(B_n)^{-1}$ is the p-adic continued fraction approximation of a p-adic number. It should be noted that $A_n(B_n)^{-1}$ can also be evaluated in a real sense; that is, if $(B_n)^{-1}$ is taken to be $\frac{1}{B_n} \in \mathbb{Q}$ and not as the p-adic inverse of B_n, then $A_n(B_n)^{-1} = \frac{A_n}{B_n}$ is a real number. These two ways of looking at an evaluated p-adic continued fraction will be of importance in section 5.1.

4.2 Period Detection

In order to test for the periodicity of continued fraction expansions, a search algorithm is required. Thus, a standard linear period search algorithm was implemented:

> Let $l = [l_1, \ldots, l_n]$ be the sequence of numbers of size n to be tested for periodicity. In the below algorithm s will represent the size of the search window and p will represent the position of the search window in the sequence l. Let s=1 and p=1.

```
[Step 1] Scan what is under the window into memory:
    let $m = [l_p, \ldots, l_{p+s-1}]$

[Step 2] Sequentially compare what was scanned to where it
    should appear again if it is a period:
    let $m_i = [l_{p+i\cdot s}, \ldots, l_{p+(i+1)\cdot s-1}]$ for $i \in \mathbb{N}$, $1 \leq i \leq \frac{n}{i}$. If
    $m = m_i$ for all $i$, then $m$ represents the period of $l$:
    quit the algorithm.

[Step 3] If the window size can be increased, increase it
    by one, otherwise change position and start again:
    if $s < \frac{n-p}{3}$, then $s = s+1$ and go to Step 2. Otherwise,
    if $p < n-2$, then $p = p+1$ and go to Step 2.
    Otherwise, we are at the end of the sequence and no
    period has been observed: quit the algorithm.
```

In searching for periods, we require at least three repetitions of the period for it to be recognized. This is implemented in step 3 when determining which variable, s or p, should be increased.

Due to the issue of precision discussed in section 3.3, this algorithm, though sound, may miss periods that are too long to replicate within its window size or periods that do not begin until very late in the sequence.

5 Periodicity Observations

5.1 Apparent Sufficient and Necessary Conditions

Algorithms I and II (see sections 3.1 and 3.2, respectively) always produce a p-adic continued fraction aproximation which converges to the number being approximated. This convergence, however, is only guaranteed in the p-adic sense. When the p-adic continued fraction approximation of $\sqrt{m}$, $m \in \mathbb{N}$, is evaluated in the real sense (see section 4.1) it sometimes convergent to the real number $\sqrt{m} \in \mathbb{R}$ and sometimes is not.

In $\mathbb{R}$, a simple continued fraction is eventually periodic if and only if it converges to a number of the form $a\sqrt{m} + b$ for some $a, b, m \in \mathbb{Q}$. In $\mathbb{Q}_p$, however, this is not the case: sometimes the p-adic continued fraction of $\sqrt{m} \in \mathbb{Q}_p$ is periodic and sometimes it is not.

When the p-adic continued fraction is produced by means of Browkin's Algorithm II (see [2]), it has been shown to always p-adically converge. That is,

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{m} \right|_p = 0 \text{ for } \sqrt{m} \in \mathbb{Q}_p,$$

regardless of whether or not it is periodic. From this point on, all references to a "p-adic continued fraction" shall be assumed to have been produced by means of Algorithm II. It has been observed that a p-adic continued fraction is also convergent in a real sense if and only if it is periodic. That is,

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{m} \right| = 0 \text{ for } \sqrt{m} \in \mathbb{R},$$

if and only if the p-adic continued fraction is periodic. This is worthy of remark since it serves as characterization of a family of sequences that converge in both the p-adic sense and in the real sense. In addition to this, the algebraic object to which these sequence converge is the same in both cases.

In summary, a p-adic continued fraction representation of $\sqrt{m} \in \mathbb{Q}_p$ displays the property that

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{m} \right|_p = 0 \text{ for } \sqrt{m} \in \mathbb{Q}_p, \text{ and } \lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{m} \right| = 0 \text{ for } \sqrt{m} \in \mathbb{R}.$$

if and only if it is periodic. This is only an observation, but it has been verified for primes $p \leq 863$ and for p-adic continued fractions of $\sqrt{m}$ for $2 \leq m \leq 150$. For a complete listing of the data, see the website indicated at the end of the references.

The following theorem will illustrate the method used to demonstrate the real convergence of a p-adic continued fraction.

Theorem 1. *If $\alpha \in \mathbb{R}$ has a periodic continued fraction representation, then $\alpha \in \mathbb{Q}(\sqrt{m})$ for some $m \in \mathbb{Q}$ with $m \geq 0$.*

Proof. Consider a non-simple periodic continued fraction of the form:

$$\alpha = \left[a_0, \ldots, a_l, \overline{b_0, \ldots, b_m}\right], \; a_i, b_i \in \mathbb{Q}.$$

Letting x represent the periodic part of α we may write,

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_l + x}}} \tag{4}$$

where x is recursively defined to be

$$x = \cfrac{1}{b_1 + \frac{1}{\cdots + \frac{1}{b_m + x}}}.$$

Since x is a finite continued fraction, it may be reduced to its simplest form, resulting in a recursive relationship of the form

$$x = \frac{ax+b}{cx+d}, \text{ for } a, b, c, d \in \mathbb{Z}. \tag{5}$$

For a complete discussion on the reducion of a finite or infinite continued fraction, see [4]. Now, (5) implies that

$$0 = cx^2 + (d-a)x - b,$$

Since α is an irrational number (an infinite continued fraction is irrational, see [4]), we know that $c \neq 0$ and $(d-a)^2 + 4cb > 0$ and is not a perfect square. Solving for x we find that

$$x \in \mathbb{Q}(\sqrt{m}) \text{ for } m = (d-a)^2 + 4cb. \tag{6}$$

Since

$$\alpha = [a_0, \ldots, a_l, x],$$

we have from (6) that $\alpha \in \mathbb{Q}(\sqrt{m})$. □

The proof of this theorem provides a simple way in which to prove that a given p-adic continued fraction converges in the real sense. Please see section 6.2 for examples of this method in use.

5.2 Periodicity Prediction

Below is presented an algorithm which is able to predict whether a p-adic number of the form $\sqrt{m}$, $m \in \mathbb{N}$ has a periodic continued fraction approximation for either Algorithm I or II. This is accomplished by evaluating the approximation in the real sense and comparing it to the real number which it should converge to. Given $\alpha = \sqrt{m} \in \mathbb{Q}_p$, $m \in \mathbb{N}$, we define the function

$$r_n : \mathbb{Q}_p \longrightarrow \mathbb{Q}.$$

Where $r_n(\alpha)$ is the p-adic continued fraction approximation of length n of $\alpha \in \mathbb{Q}_p$ evaluated in the real sense (see section 4.1). Formally, for a finite or infinite p-adic continued fraction approximation of α,

$$[b_0, b_1, \ldots, b_n, \ldots], \text{ let } r_n(\alpha) = \frac{A_n}{B_n} \in \mathbb{Q},$$

for $\frac{A_n}{B_n}$ defined as in (3). With this new function, we now define a new algorithm:

Given $\alpha \in \mathbb{Q}_p$, let $i = 1$.

```
[Step 1] $i = i + 1$.
[Step 2] If $|r_{i-1}(\alpha) - r_i(\alpha)| \geq tol$, then go to Step 1.
[Step 3] If $|\alpha - r_i(\alpha)| < tol$ then the $p$-adic continued
    fraction expansion of $\alpha$ given by Algorithm I or II is
    periodic.  Quit the algorithm.
```

Where $|\cdot|$ is the standard absolute value and *tol* is the number of digits of decimal precision required to force algorithm to make a decision.

Where *tol* is how much error is to be allowed for a sequence to be considered convergent. If *tol* is set too close to zero, the efficiency of the function will decrease. If *tol* is set too high, then there will be many false positives. In all cases, however, this function will run significantly faster than Algorithm I or II when using an equivalent level of p-adic precision, as discussed in section 3.3.

6 Data

6.1 Verification of Conditions

The sufficient and necessary conditions for the periodicity of a p-adic continued fraction given in section 5.1 was verified for p-adic numbers of the form $\sqrt{m}$ for $2 < m \leq 150$ and for primes $p \leq 863$. The verification took a total time of 297 hours, 41 minutes, and 5 seconds. Since the data gathered from this is of a large volume, typical results are given below with the complete results available online (see end of references for website). The results for Algorithm II displayed a greater number of periodic p-adic continued fractions then Algorithm I, thus the following results are for Algorithm II; the results for Algorithm I are extremely similar.

The following two examples of results are typical of the general data.

6.1.1 $\mathbb{Q}_{19}$

```
sqrt(5): 4.999999999999999999999999999. periodic.
sqrt(6): 5.999999999999999999999999999. periodic.
sqrt(7): 166.6907868099887408379569838. not periodic.
sqrt(11): 25.61836864115646697971399229. not periodic.
sqrt(17): 17.00000000390724321569231962. periodic.
sqrt(20): 19.99999999999999999999999999. periodic.
sqrt(23): 23.00000000000000000000000000. periodic.
```

```
sqrt(24): 23.99999999999999999999999999. periodic.
sqrt(26): 40.04740916430207916906667074. not periodic.
sqrt(28): 27.99999999999999999999999999. periodic.
sqrt(30): 29.99999999999999999999999999. periodic.
sqrt(35): 34.99999999999999999999999999. periodic.
sqrt(39): 38.99999999999999999999999999. periodic.
sqrt(42): 41.99999999999999999999999999. periodic.
sqrt(43): 42.99999999999999999999999999. periodic.
sqrt(44): 14.71786900878706822479419327. not periodic.
sqrt(45): 44.99999999999999999999999999. periodic.
sqrt(47): 46.99999999999999999999999999. periodic.
sqrt(54): 53.99999999999999999999999999. periodic.
sqrt(55): 23.01484459432193820865505545. not periodic.
sqrt(58): 9.797839403953568372936695464. not periodic.
sqrt(61): 2.917758165715961155807893019. not periodic.
sqrt(62): 61.99999999999999999999999999. periodic.
sqrt(63): 62.99999999999999999999999999. periodic.
sqrt(66): 65.99999999999999999999999999. periodic.
sqrt(68): 67.99999999999999999999999999. periodic.
sqrt(73): 18.93619733733758143712169992. not periodic.
sqrt(74): 73.99999999999999999999999999. periodic.
sqrt(77): 1.309240183953138248185576792. not periodic.
sqrt(80): 79.99999999999999999999999999. periodic.
sqrt(82): 6.189845729656622245525783134. not periodic.
sqrt(83): 16.29146922823558046800251607. not periodic.
sqrt(85): 0.8592932319083533035799076278. not periodic.
sqrt(87): 86.99999999999999999999999999. periodic.
sqrt(92): 92.00000000000000000000000000. periodic.
sqrt(93): 92.99999999999998256900507000. periodic.
sqrt(96)): 95.99999999999999999999999999. periodic.
sqrt(99)): 1.536158563056778847660864568. not periodic.
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6.1.2 $\mathbb{Q}_{59}$

```
sqrt(3): 3.000000000000000000000000000. periodic.
sqrt(5): 4.999999999999999999999999999. periodic.
sqrt(7): 284.2213999520526848549290210. not periodic.
sqrt(12): 586.9721278971492235224642271. not periodic.
sqrt(15): 702.3017469815094084244298821. not periodic.
sqrt(17): 666.1559449131931943263339620. not periodic.
sqrt(19): 150.5574027854119384626045078. not periodic.
sqrt(20): 111.4258850696768646255269061. not periodic.
sqrt(21): 837.0823735185217983003738990. not periodic.
sqrt(22): 21.99999999999999999999999999. periodic.
sqrt(26): 25.99999999999999999999999999. periodic.
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```
sqrt(27): 1024.882176845683781144233070. not periodic.
sqrt(28): 257.1926421181337340037456655. not periodic.
sqrt(29): 238.8073271650882070520741898. not periodic.
sqrt(35): 446.2458387159500433407197784. not periodic.
sqrt(41): 41.00000000000000000000000000. periodic.
sqrt(45): 483.1412777392206103218303041. not periodic.
sqrt(46): 672.8761872578711261746684370. not periodic.
sqrt(48): 93.39026187295283244275356333. not periodic.
sqrt(51): 50.99999999999999999999999999. periodic.
sqrt(53): 377.5855864920883101565380483. not periodic.
sqrt(57): 423.3738594366371831279586550. not periodic.
sqrt(60): 59.99999999999999999999999999. periodic.
sqrt(62): 61.99999999999999999999999999. periodic.
sqrt(63): 62.99999999999999999999999999. periodic.
sqrt(66): 247.4585047028202301624143923. not periodic.
sqrt(68): 67.99999999999999999999999999. periodic.
sqrt(71): 607.5249003353226013530119752. not periodic.
sqrt(74): 456.4722241599938680209390893. not periodic.
sqrt(75): 74.99999999999999999999999999. periodic.
sqrt(76): 991.3451657825577173170353943. not periodic.
sqrt(78): 78.00000000000000000000000000. periodic.
sqrt(79): 392.6574396938294086835373115. not periodic.
sqrt(80): 80.11976553220414239566268897. periodic.
sqrt(84): 84.00000000000000000000000000. periodic.
sqrt(85): 84.99999999999999999999999999. periodic.
sqrt(86): 538.1002487701451775463186964. not periodic.
sqrt(87): 253.6815295433979868223502795. not periodic.
sqrt(88): 163.7452484907808647987065467. not periodic.
sqrt(94): 2653.070745136395836339617359. not periodic.
sqrt(95): 94.99999999999999999999999999. periodic.
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6.2 Verification of Specific Fractions

In each of the following examples a periodic p-adic continued fraction for $\sqrt{m} \in \mathbb{Q}_p$, $m \in \mathbb{N}$, given by Browkin's Algorithm II (see [2]) will be presented and shown to converge in the real sense to $\sqrt{m} \in \mathbb{R}$. This data represents a small sample of the total amount of data, which has been verified for primes $p \leq 863$ and $2 \leq m \leq 150$ in $\sqrt{m} \in \mathbb{Q}_p$. For a complete listing of the data, see the website indicated at the end of the references.

6.2.1 The case of $m = 7$, $p = 3$

Let α be the p-adic continued fraction representation of $\sqrt{7} \in \mathbb{Q}_3$ generated by Algorithm II:

$$\alpha = \left[1, \overline{1/3, -1, -1/3, 1, -1/3, 1}\right].$$

α has been shown to converge p-adically to $\sqrt{7} \in \mathbb{Q}_3$. However, we would also like to show that α converges in the real sense:

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{7} \right| = 0 \text{ for } \sqrt{7} \in \mathbb{R}.$$

Since α is periodic we can write

$$\alpha = 1 + x \text{ with } x = [1/3, -1, -1/3, 1, -1/3, 1, x].$$

Reducing x via the standard algorithm results in the recursive relationship

$$x = \frac{33x - 24}{-4x + 25}. \tag{7}$$

Solving for x in (7) we find that $x = -1 \pm \sqrt{7}$. Since α is positive, we take the positive case of x:

$$x = -1 + \sqrt{7},$$

and so from above we have that

$$\alpha = \sqrt{7} \in \mathbb{R}.$$

Remarkably, α also converges p-adically to $\sqrt{7} \in \mathbb{Q}_3$; that is

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{7} \right|_p = 0 \text{ for } \sqrt{7} \in \mathbb{Q}_3.$$

So α is a continued fraction representation of $\sqrt{7} \in \mathbb{R}$ as well as a p-adic continued fraction representation of $\sqrt{7} \in \mathbb{Q}_3$.

6.2.2 The case of $m = 6$, $p = 5$

Let α be the p-adic continued fraction representation of $\sqrt{6} \in \mathbb{Q}_5$ generated by Algorithm II:

$$\alpha = \left[1, \overline{2/5, 2}\right].$$

α has been shown to converge p-adically to $\sqrt{6} \in \mathbb{Q}_5$. However, we would also like to show that α converges in the real sense:

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{6} \right| = 0 \text{ for } \sqrt{6} \in \mathbb{R}.$$

Since α is periodic we can write

$$\alpha = 1 + x \text{ with } x = [2/5, 2, x].$$

Reducing x via the standard algorithm results in the recursive relationship

$$x = \frac{5x + 10}{2x + 9}. \tag{8}$$

Solving for x in (8) we find that $x = -1 \pm \sqrt{6}$. Since α is positive, we take the positive case of x:

$$x = -1 + \sqrt{6},$$

and so from above we have that

$$\alpha = \sqrt{6} \in \mathbb{R}.$$

Remarkably, α also converges p-adically to $\sqrt{6} \in \mathbb{Q}_5$; that is

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{6} \right|_p = 0 \text{ for } \sqrt{6} \in \mathbb{Q}_5.$$

So α is a continued fraction representation of $\sqrt{6} \in \mathbb{R}$ as well as a p-adic continued fraction representation of $\sqrt{6} \in \mathbb{Q}_5$.

6.2.3 The case of $m = 2$, $p = 7$, I

Let α be the p-adic continued fraction representation of $\sqrt{2} \in \mathbb{Q}_7$ generated by Algorithm II:

$$\alpha = \left[3, \overline{1/7, 3, -20/49, 3, 1/7, -1, 4/7, -1}\right].$$

α has been shown to converge p-adically to $\sqrt{2} \in \mathbb{Q}_7$. However, we would also like to show that α converges in the real sense:

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{2} \right| = 0 \text{ for } \sqrt{2} \in \mathbb{R}.$$

Since α is periodic we can write

$$\alpha = 3 + x \text{ with } x = [1/7, 3, \ldots, -1, x].$$

Reducing x via the standard algorithm results in the recursive relationship

$$x = \frac{23121x + 14168}{-2024x + 10977}. \tag{9}$$

Solving for x in (9) we find that $x = -3 \pm \sqrt{2}$. Since α is positive, we take the positive case of x:

$$x = -3 + \sqrt{2},$$

and so from above we have that

$$\alpha = \sqrt{2} \in \mathbb{R}.$$

Remarkably, α also converges p-adically to $\sqrt{2} \in \mathbb{Q}_7$; that is

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{2} \right|_p = 0 \text{ for } \sqrt{2} \in \mathbb{Q}_7.$$

So α is a continued fraction representation of $\sqrt{2} \in \mathbb{R}$ as well as a p-adic continued fraction representation of $\sqrt{2} \in \mathbb{Q}_7$.

6.2.4 The case of $m = 2$, $p = 79$, II

Let α be the p-adic continued fraction representation of $\sqrt{2} \in \mathbb{Q}_{79}$ generated by Algorithm II:

$$\alpha = \left[9, \overline{-18/79, 18}\right].$$

α has been shown to converge p-adically to $\sqrt{2} \in \mathbb{Q}_{79}$. However, we would also like to show that α converges in the real sense:

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{2} \right| = 0 \text{ for } \sqrt{2} \in \mathbb{R}.$$

Reducing x via the standard algorithm results in the recursive relationship
Since α is periodic we can write

$$\alpha = 9 + x \text{ with } x = [-18/79, 18, x]. \tag{10}$$

Reducing x via the standard algorithm results in the recursive relationship

$$x = \frac{79x + 1422}{-18x - 245}. \tag{11}$$

Solving for x in (11) we find that $x = -9 \pm \sqrt{2}$. Since α is positive, we take the positive case of x:

$$x = -9 + \sqrt{2},$$

and so from above we have that

$$\alpha = \sqrt{2} \in \mathbb{R}.$$

Remarkably, α also converges p-adically to $\sqrt{2} \in \mathbb{Q}_{79}$; that is

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{2} \right|_p = 0 \text{ for } \sqrt{2} \in \mathbb{Q}_{79}.$$

So α is a continued fraction representation of $\sqrt{2} \in \mathbb{R}$ as well as a p-adic continued fraction representation of $\sqrt{2} \in \mathbb{Q}_{79}$.

6.2.5 The case of $m = 26$, $p = 229$

Let α be the p-adic continued fraction representation of $\sqrt{26} \in \mathbb{Q}_{229}$ generated by Algorithm II:

$$\alpha = \left[22, \overline{-22/229, 44} \right].$$

α has been shown to converge p-adically to $\sqrt{26} \in \mathbb{Q}_{229}$. However, we would also like to show that α converges in the real sense:

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{26} \right| = 0 \text{ for } \sqrt{26} \in \mathbb{R}.$$

Since α is periodic we can write

$$\alpha = 22 + x \text{ with } x = [-22/229, 44, x].$$

Reducing x via the standard algorithm results in the recursive relationship

$$x = \frac{229x + 10076}{-22x - 739}. \tag{12}$$

Solving for x in (12) we find that $x = -22 \pm \sqrt{26}$. Since α is positive, we take the positive case of x:

$$x = -22 + \sqrt{26},$$

and so from above we have that

$$\alpha = \sqrt{26} \in \mathbb{R}.$$

Remarkably, α also converges p-adically to $\sqrt{26} \in \mathbb{Q}_{229}$; that is

$$\lim_{n\to\infty} \left| \frac{A_n}{B_n} - \sqrt{26} \right|_p = 0 \text{ for } \sqrt{26} \in \mathbb{Q}_{229}.$$

So α is a continued fraction representation of $\sqrt{26} \in \mathbb{R}$ as well as a p-adic continued fraction representation of $\sqrt{26} \in \mathbb{Q}_{229}$.

7 Thanks

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References

[1] Browkin, Jerzy. “Continued Fractions in Local Fields, I”. *Demonstratio Math.* Vol. 11 (1978): 67-82.

[2] Browkin, Jerzy. “Continued Fractions in Local Fields, II”. *Mathematics of Computation.* Vol. 70, No. 235: 1281-1292.

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[5] Kaplansky, Irving. *Set theory and metric spaces.* Allyn and Bacon: 1972.

[6] Koblitz, Neal. *P-adic Numbers, P-adic Analysis, and Zeta-Functions.* New York : Springer-Verlag: Graduate Texts in Mathematics: 1977.

Extended results, code, and this report can be found at:

`http://www.math.arizona.edu/~mmoore/padic_cont_frac.`