Geometric Aspects of the Heisenberg Group

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Abstract

I provide a background of groups viewed as metric spaces to introduce the notion of asymptotic dimension of a group. I analyze the asymptotic dimension of $\mathbb{Z} \oplus \mathbb{Z}$ and the free group on two generators to better understand the concept of asymptotic dimension. The asymptotic dimension of the Heisenberg group, $\mathbb{H}$, $\operatorname{asdim} \mathbb{H}$, has been shown to be three using advanced mathematics. I will try to show that $\operatorname{asdim} \mathbb{H}$ is three and create a cover for $\mathbb{H}$ which will confirm the result about the asymptotic dimension. Also, the Cayley graph of $\mathbb{H}$ and the algebraic structure of $\mathbb{H}$ is analyzed to help understand this group better.

1 Introduction

When studying groups, the question arises of how to geometrically describe groups. The typical and most logical way to do this is with Cayley graphs. Once a Cayley graph is constructed, we can look for interesting properties that arise from the groups construction. A notion that arises is the idea of asymptotic dimension for infinite discrete groups. To understand this concept, a little background in metric spaces, group theory, and topology is needed. This project was done in collaboration with Roeland Hancock under the supervision of Dorin Dumitrascu with funding from the Undergraduate Research Assistant program at the University of Arizona.

1.1 Metric Spaces

Roughly stated, a *metric space* is any set of elements where some distance between the elements can be measured. Formally stated, a metric space is a set X such that for any two points x and y in X, there is a nonnegative real number called the *distance* between x and y. The distance can be thought of as a mapping, $d : X \times X \to \mathbb{R}^+$. For a set to be a metric space (X, d) there must be a distance mapping which satisfies the following four properties:

1. (non-negativity) $d(x, y) \geq 0$
2. (identity) $d(x, y) = 0 \iff x = y$
3. (symmetry) $d(x, y) = d(y, x)$
4. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$for all z in X.[Fi]

Some common metric spaces are $\mathbb{R}$ with $d(x, y) = |x - y|$. This is not the only possibility for the distance mapping, since $d_c(x, y) = c|x - y|$ where $c \in (0, \infty)$ also satisfies all four properties. Other metric spaces are $\mathbb{R}^n$ with some possible distance mappings: $d_1(x, y) = \sum_{i=1}^n |x_i|$, $d_2(x, y) = \sqrt{\sum_{i=1}^n x_i^2}$, or $d_\infty(x, y) = max_{1 \leq i \leq n} |x_i|$. Another common metric space is the set of all continuous functions over a closed interval, $C([a, b],\ \mathbb{R}) = \{f : [a, b] \to \mathbb{R} : f\ \textit{is continuous}\}$ with the distance mapping
$d(f, g) = sup\, \{|f(x) - g(x)| : x \in [a, b]\}$[Fa].

1.2 Groups as Metric Spaces

I am primarily concerned with infinite discrete groups as metric spaces. At first it may seem unusual to think of groups as a metric space, but actually groups can be viewed very naturally as a metric space because the construction of groups. For a group G, let S be a set of generators that is symmetric, if $s \in S$ then $s^{-1} \in S$. We will call elements of S *letters* and elements of G *words*, then for every word g ∈ G define the *length* of g, $l(g)$, to be the minimal number of letters that are needed to construct g. Notice that $l(e) = 0$ and if $s \in S$ then $l(s) = 1$. Define the *word metric* $d(x, y) = l(xy^{-1})$[Bl], it can be easily checked that this mapping satisfies all four of the properties necessary for G to be a metric space. It is important to note that the length of a word depends on the generating set. For example in the group $\mathbb{Z} \oplus \mathbb{Z}$ if $S = \{(1,0),(0,1),(-1,0),(0,-1)\}$then $l(1,1) = 2$, but if $S = \{(1,1),(0,1),(-1,-1),(0,-1)\}$ then $l(1,1) = 1$.

Since groups can be viewed as metric spaces, it leads us to wonder if there is any geometry in these groups. The most common way to picture a group is through its *Cayley graph*. For a finitely generated group G with a finite generating set S the Cayley graph consists of vertices and edges. The elements of G are the vertices and an edge connects two vertices x and y if xy^{-1} is in S[Ga]. This construction allows us create and visualize groups and explore geometric properties. Elements of G that have an edge between them in the Cayley graph have word distance one, and similarly the word distance between any two elements x and y is the minimal number of edges from x to y.

2 Heisenberg Group

The *discrete Heisenberg group* is defined:

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,\, y,\, z \in \mathbb{Z} \right\}$$

under the operation of the usual matrix multiplication. Throughout this paper, we shall let

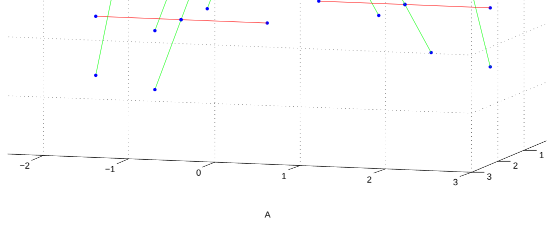
$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

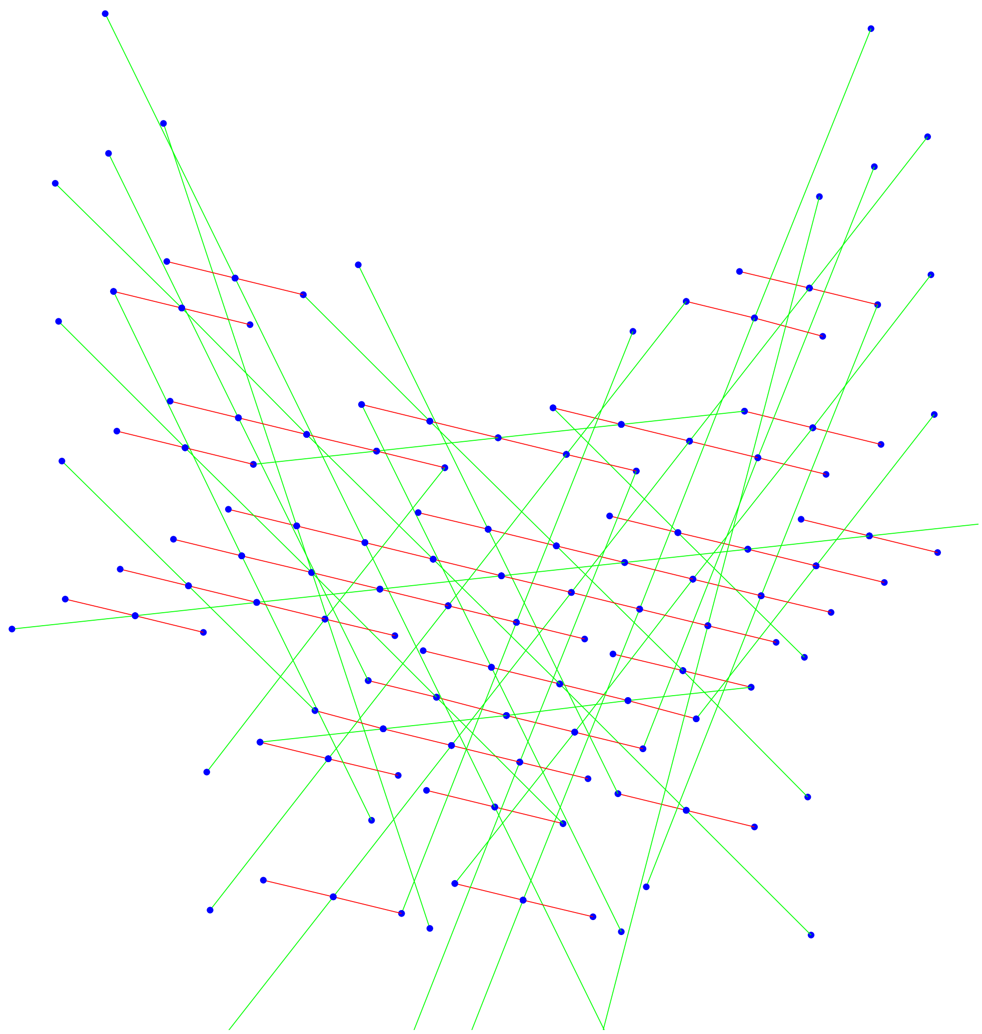
The standard generating set for the Heisenberg group is $\{A, B, A^{-1}, B^{-1}\}$, and the group has the relation

$$ABA^{-1}B^{-1} = BA^{-1}B^{-1}A = A^{-1}B^{-1}AB = B^{-1}ABA^{-1} = C.$$

Unless otherwise specified, this is the generating set that will be used when computing distances. Another generating set is $\{A, B, C, A^{-1}, B^{-1}, C^{-1}\}$ which can also be used to compute distances, but results a different metric. For example using the first generating set $l(C) = 4$, while using the second generating set $l(C) = 1$. Also the Cayley graphs that represent $\mathbb{H}$ are very different with respect to different generating sets. For example, the second generating set has six edges leaving each vertex while the first generating set only has four edges leaving each vertex.

A more convenient way than matrices to denote elements in this group is by three-tuples of numbers. Using this notation, $\mathbb{H} = \{(x, y, z) : x,\ y,\ z\ \in \mathbb{Z}\}$, and the multiplicative operation of elements can be written:
$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy')$.
This notation makes computations in the Heisenberg group much more efficient. For example $A = (1, 0, 0),\ B = (0, 1, 0),\ and\ AB = (1, 1, 1)$. It is important to note that $(x, y, z)^{-1} = (-x, -y, -z+xy)$, and the identity of $\mathbb{H}$, $I_3 = (0, 0, 0)$. Since elements of $\mathbb{H}$ can be written in three-tuples of numbers, (x, y, z), it is natural to think of its Cayley graph embedded in $\mathbb{Z}^3 \subseteq \mathbb{R}^3$. If the Cayley graph is drawn this way, it can be tempting to use the Euclidean norm, but is wrong for obvious reasons. Here is a computer illustration of a ball of radius three and four in the Heisenberg group centered at the identity using this embedding. The program used to generate these images was created by Roeland Hancock. The red lines represent A and the blue lines represent B. The ball of radius three is oriented with the positive A axis facing right, the positive B axis facing the reader, and the positive C axis facing up. The ball of radius four is oriented with with the positive A axis facing out and to the right, the positive B axis facing back an to the right, and the positive C axis facing up.





3 Asymptotic Dimension

When thinking about geometry, one of the most basic concepts is dimension. A line has dimension one, a plane has dimension two, a cube has dimension three, and so on. This concept refers to continuous shapes, so it will not work to describe the dimension of discrete objects like finitely generated groups. In linear algebra the definition of dimension refers to a vector space over a field[FIS], and the dimension is the cardinality of a maximally linearly independent set. But the majority of groups are not vector spaces, and so this definition will not work either. Usually geometry does not deal with discrete sets, but with the correct perspective some discrete sets can appear continuous. Imagine the integers on the real line, as we move further and further away the elements get closer and closer together and eventually appear to be a line. This observation and other geometric considerations lead Gromov to introduce the concept of *asymptotic dimension*[Gr].

Definition 3.1.

The ***asymptotic dimension*** of a metric space (X, d), asdim(X), is the smallest counting number n such that one of the following equivalent statements hold:

1. For any $D > 0$ there exist a *uniformly bounded cover* $C = \{U_i : i \in I\}$ such that no *ball* of radius D in X intersects more than $(n+1)$ members of the cover C.

2. For any $R > 0, \exists$ *D-disjoint* families $U_0, U_1, ..., U_n$ of uniformly bounded sets whose union covers X.[BD1]

3. For any $L > 0 \exists$ uniformly bounded cover X with multiplicity $\leq n + 1$ and with *Lebesque number* $> L$.

4. For any $\epsilon > 0 \exists$ *uniformly cobounded*, ϵ-Lipschitz map $\phi : X \to K$ to a *uniform polyhedron* of dim$\leq n$.[BD2] Where:

Here is a dictionary for the above statements:

- A *cover* of X is a collection of sets, U_i, such that $X \subset \cup_{i \in I} U_i$, where I is an indexing set.

- A cover is *uniformly bounded* if there exists a positive number M, such that the cardinality of every member of the cover is less than M.

- A *ball* of radius D, centered at x in a metric space X is $B(x,D) = B_D(x) = \{y \in X : d(x,y) \leq D\}$

- Two sets U^1, U^2 *with* $U^1 \neq U^2$ are D-disjoint if $inf\{d(x_1,x_2) : x_1 \in U^1, x_2 \in U_2\} \geq d$

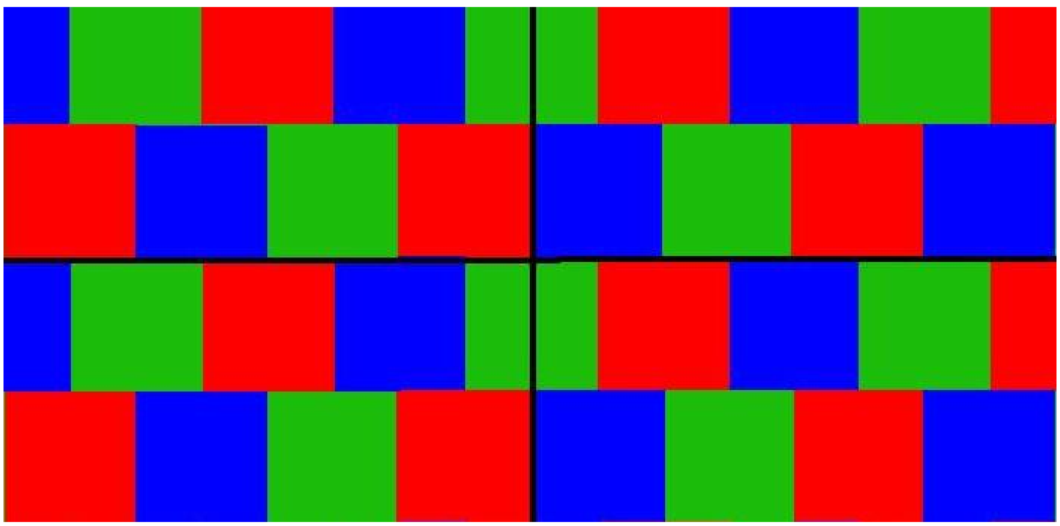
- The *Lebesque number* of a cover U of metric X is $L(U) = inf\{max\{d(x, X/U_i) : U_i \in U\} : \ x \in X\}$

- A map ϕ to a uniform polyhedron is *uniformly cobounded* if there exists B such that diam$(\phi^{-1}(\sigma)) \leq B$, for all simplices σ.

- *Uniform polyhedron* is a geometric realization of a simplicial complex in l^2, the metric space of all square summable infinite series of Real numbers, with the metric it inherits as a subset.

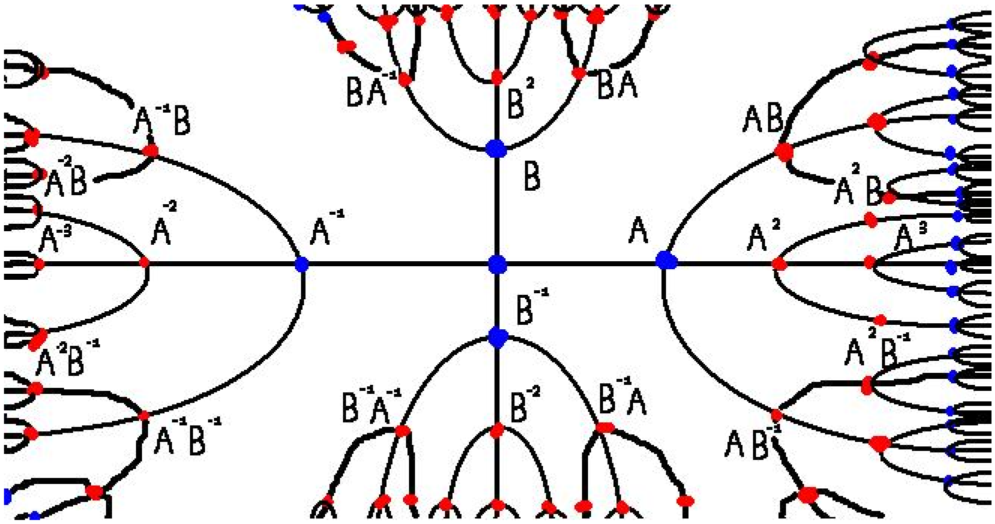
Only 1 and 2 will be used in this paper, but 3 and 4 are included for completeness. To show equivalence of the previous statements I will first show that the first definition implies the second definition: assume that there exists a cover of sets such that for all x in X, $B_D(x)$ intersects at most n+1 members of the cover. So the sets can be grouped into n+1 families, and with the possible relabeling of the sets each family will have to be at least D-disjoint. If the families were not d-disjoint, then there would be a ball that intersects more that n+1 sets. The second and the third are identical because being D-disjoint and having a Lebesque number>L is the same concept but expressed slightly different. Changing L to 2L or D to 2D might be necessary to make the correct amount of separation between the sets, but since it is for any D and for any L, it is not needed. The fourth definition is a more geometric definition which will not be shown equivalence but again is included for completeness. To complete the equivalence the second implies the first: assume a cover is made of uniformly bounded families of d-disjoint sets. Then for all x in X, $B_D(x)$ can only intersect n+1 sets because the families of sets are at least distance d apart, which prevents $B_D(x)$ from intersecting two sets from the same family.

It is interesting that $\mathbb{Z}$ has asymptotic dimension one, similar to a line which has regular dimension one. This can be seen since for $R > 0$ we can create a cover which contains intervals of length $R + 1$ and two sets in the cover may only touch at the endpoints. A little harder to see but still simply constructed, $\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ has asymptotic dimension two. For $R > 0$ create a cover which contains sets of side length $2R + 1$ that only touch at the edges. So using the second definition of asymptotic dimension, it is clear that all of the sets of the same color are at least R-disjoint. This is a possible cover to show that asdim $\mathbb{Z}^2 = 2$.



Generally it can be proved that $\mathbb{Z}^n$ has asymptotic dimension n. This group is relatively easy to study because it is an *Abelian* group, $ab = ba$ for all a,b in G. It is also nice because every part of the Cayley graph has identical structure. Also notice that any finite metric space has dimension zero, because the entire metric space can be in one set that is bounded by the order of the group.

The free group on two generators, $\mathbb{F}_2$, has no relation between generators. With a little bit more work, it can be seen that asdim $\mathbb{F}_2$ is one. To construct a cover, start with the identity and let the first set be the ball of radius R. The next sets will start from elements on the outer edge of the previous ball and will include all elements between the starting element and every element that is less than a distance of $3R$ from the starting element. Repeating this process will construct a cover which defines asdim $\mathbb{F}_2$ to be one. This same process can be applied to construct a cover for a free group on n elements, and thus asdim $\mathbb{F}_n$ is one. This is a drawing of covers in $\mathbb{F}_2$ for $R = 1$.



4 Asymptotic Dimension of the Heisenberg Group

To determine the asymptotic dimension of the Heisenberg Group, I will first try to understand the structure of the group. To begin it is helpful to set some of the variables to zero. Recall the product for $\mathbb{H}$:

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy')$$

Notice that if either x and x' or y and y' are both zero, then the new product is $(x, 0, z) \cdot (x', 0, z') = (x + x', 0, z + z')$ or $(0, y, z) \cdot (0, y', z') = (0, y + y', z + z')$. In both cases ignoring the zero component results in this set of elements, $\gamma = \{(x, 0, z) : x, z \in \mathbb{Z}\}$, has the same operation as that in $\mathbb{Z}^2$. For γ to be a subgroup of the $\mathbb{H}$, the identity must be in γ, and it clearly is. Also γ must be closed under multiplication by inverses, a simple calculation: $(x, 0, z) \cdot (x', o, z')^{-1} = (x, 0, z) \cdot (-x', 0, -z') = (x - x', 0, z - z')$, shows that γ is indeed closed under multiplication and γ is a subgroup of $\mathbb{H}$.

Proposition 4.1. *The* asdim *of a group,* G*, is well-defined without reference to a generating set,* S*.*

Proof. Assume S_1 and S_2 are two generating sets for G. Using the first definition of asymptotic dimension, fix $D_1 < 0$ and consider $B(x, D_1)$ generated by

S_1. Since S_2 also generates G, for some $D_2 > 0$, $B(x, D_2)$ generated by S_2 will contain $B(x, D_1)$ since every ball is bounded. Then for $D = max\{D_1, D_2\}$, a cover can be constructed using S_1such that no ball intersects more than n+1 members of the cover and a cover can be constructed using S_2satisfying the same requirements. So the group has asymptotic dimension n, regardless of the generating set S □

Theorem 4.2. *If S is a subgroup of X, a metric space, then*

$$\operatorname{asdim} S \leq \operatorname{asdim} X.$$

Proof. Assume that $S \leq X$ a proper subgroup $S \neq X$ (otherwise it would be trivial), and asdimS = n. Using the second definition of asymptotic dimension, there exists n+1 uniformly D-disjoints family of sets whose union covers S. When looking at all of X, this decrease the number of elements. Because the elements of S are elements of X and they can only be covered by n+1 families of D-disjoint sets, clearly a larger set X cannot by a smaller number of families of D-disjoint sets. So asdimS $\leq$ asdimX. □

Lets look at how this applies to the Heisenberg group. Consider $\gamma = \{(x, 0, z) : x, z \in \mathbb{Z}\} \approx \mathbb{Z}^2 \subseteq \mathbb{H}$ to show that asdim $\mathbb{H}$ is greater than or equal to two. We showed earlier that this group is isomorphic to $\mathbb{Z}^2$, and we also showed that asdim $\mathbb{Z}^2$ is two. Since this set is a subgroup of $\mathbb{H}$, then we have asdim $\mathbb{H} \geq 2$.

Theorem 4.3. *Let $\phi : G \to H$ be an epimorphism of a finitely generated group G with kernel $ker\phi = K$. Assume that* $\operatorname{asdim} K \leq k$ *and asymptotic dimension $H \leq n$. Then asymptotic dimension $G \leq n + k$[DS].*

Recall a subgroup $N \leq G$ is a normal subgroup $N \trianglelefteq G$ if $xN = Nx$. This is checked by seeing is $xnx^{-1} \in N$ for all $x \in G$ and for all $n \in N$[DS]. Recall γ which is a subgroup of $\mathbb{H}$ is also a normal subgroup of $\mathbb{H}$ which is checked by the calculation:

$$(x, y, z) \cdot (x', 0, z') \cdot (x, y, z)^{-1} =$$

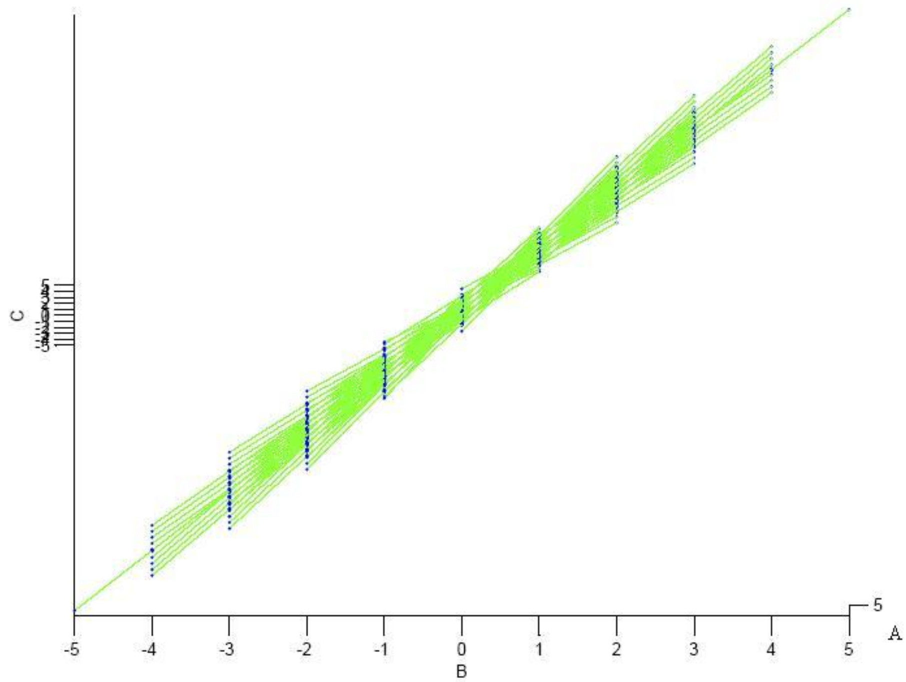
$$= (x + x', y, z + z') \cdot (-x, -y, -z + xy)$$
$$= (x', 0, z' + xy + (x + x')(-y))$$
$$= (x', 0, z' - x'y) \in \gamma.$$

So the homomorphism $\phi : \mathbb{H} \rightarrow \mathbb{H}/\gamma$ exists and is defined by $\phi(x, y, z) = (x, y, z) \cdot \gamma$ where $\gamma = ker\phi$. The image of $\mathbb{H}$, $\phi(\mathbb{H})$, is $(0, y, 0) \cdot ker\phi \approx \mathbb{Z}$ which has asymptotic dimension one. Because $\gamma \approx \mathbb{Z}^2$, $\operatorname{asdim} \gamma = ker\phi$ is two, so by Theorem 4.3, the asymptotic dimension $\mathbb{H} \leq 1 + 2 = 3$. So we have an upper bound, and $\operatorname{asdim} \mathbb{H}$ is either two or three.

Theorem 4.4. *Let G be a finitely generated polycyclic group, then* $\operatorname{asdim} G \leq h(G)$*. Where G is polycyclic if there exists a sequence of subgroups $\{e\} = G_0 \subset G_1 \subset ... \subset G_n = G$ such that $G_i \triangleleft G_{i+1}$ and the quotient group G_{i+1}/G_i is cyclic. The Hirsch Length of G, $h(G)$, is the number of factors G_{i+1}/G_i isomorphic to $\mathbb{Z}$[Ro].*

Another way to show that $\operatorname{asdim} \mathbb{H}$ is less than three is by applying the above theorem. For $\mathbb{H}$, consider the sequence of subgroups $\{e\} \subset \{(x, 0, 0) : x \in \mathbb{Z}\} \subset \{(x, 0, z) : x, z \in \mathbb{Z}\} = \gamma \subset \mathbb{H}$. The subgroup $\{(x, 0, 0) : x \in \mathbb{Z}\}$, which I will denote ζ is isomorphic to $\mathbb{Z}$ by from the definition of the operation of $\mathbb{H}$. Clearly $\{e\} \triangleleft \zeta$ and also $\zeta/\{e\} \approx \zeta \approx \mathbb{Z}$ so it is cyclic. It is easy to check that $\gamma \triangleleft \zeta$ and note that $\gamma/\zeta = \{(0, 0, z) \cdot \zeta : z \in \mathbb{Z}\}$which is isomorphic to $\mathbb{Z}$ so is cyclic. I have already shown that $\gamma \triangleleft \mathbb{H}$ and $\mathbb{H}/\gamma = \{(0, y, 0)\mathbb{H} : y \in \mathbb{Z}\}$ which also is isomorphic to $\mathbb{Z}$ and is cyclic. Therefore the Hirsch length of $\mathbb{H}$, h($\mathbb{H}$), number of factors G_{i+1}/G_i isomorphic to $\mathbb{Z}$ is 3. So from Theorem 4.4 we have shown asdim the Heisenberg group $\leq$3.

Nearly all theorems and even the definitions of asymptotic dimension give an upperbound for asdim a group. Because of this, we decided to shift our focus to construct an optimal cover for $\mathbb{H}$ that would minimize the asymptotic dimension. To simplify construction, let the generating set be the standard generating set for $\mathbb{H}$, $\{A, A^{-1,}B, B^{-1}\}$. Let the center of a ball be the point (x, y, z). When we multiply by A, $(x, y, z)(1, 0, 0) = (x + 1, y, z)$ the only value that changes is the x component increases by one. When we multiply by B, $(x, y, z)(0, 1, 0) = (x, y + 1, z + x)$, the y component increases by one and the z component increases by the value of x. Because of this, the shape of the ball when embedded in $\mathbb{Z}^3$ depends only on the x component of the center. The following pictures show a ball centered at (10,0,0) and (100,0,0) respectively. Both picture are oriented with the positive A axis facing out to the reader, the positive B axis to the right, and the positive C axis facing up. The scale on the C axis in not the same for both pictures. For the first picture the scale is [-50,50] and for the second picture the scale on the C axis is [-500,500].


C
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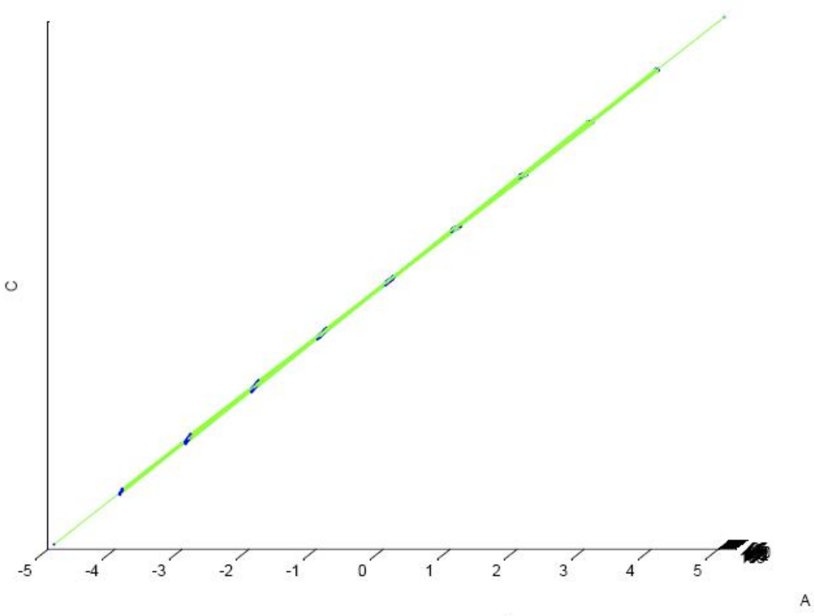
C
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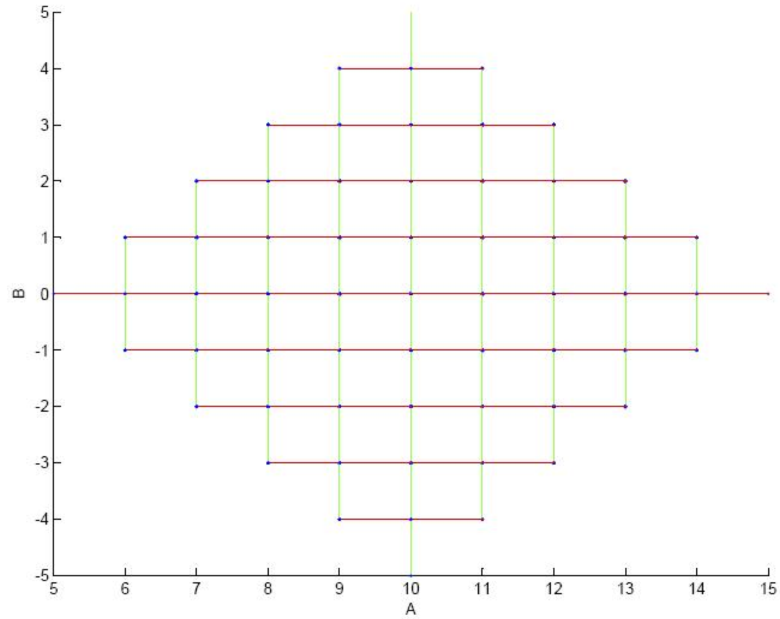
The best cover that I found only restricted the asymptotic dimension of $\mathbb{H}$ to be less than or equal to five. The cover is constructed using blocks shaped like parallelepipeds. For a fixed $R > 0$ each parallelepiped is constructed in the following way. along the x or A direction the parallelepiped will be parallel to the x- axis and have a width of 2R. This is because multiplying by A it only changes the x component, just like with $\mathbb{Z}^3$. Along the z or C direction, the parallelepiped will be parallel to the C axis and have a width of 2R. Along the B direction, since multiplying by B changes both the y value and the z value, the parallelepiped will be slanted according to the x value at the time B is multiplied. Since the block is 2R wide along the x-axis, the block will incorporate different x values. Starting at $(x - R, y, z)$ multiply by B^R and B^{-R} this will end on $(x - R, y + R, z + (x - R)R)$ and $(x - R, y - R, z - (x - R)R)$ respectively, from these two points go up R and down R. This will give us four corners of the parellelepiped. Repeat the same process starting with $(x + R, y, z)$ and this will give the other corners of the parallelepiped. So the set in the cover is the integer values of this parallelepiped.

For a fixed x, tile the corresponding yz plane in a fashion similar to $\mathbb{Z}^2$ and we will get three families of R-disjoint sets. Next, repeat the process at the point $(x+2R, y, z)$ and we will get three more families of R-disjoint sets. These sets will not be R-disjoint from the first family of sets, but will be right next to them. Repeating the process at $(x + 4R, y, z)$ will result in sets greater than R-disjoint from the first three families of sets, so they can be labeled to be part of the first three families of R-disjoint sets. This process determines our 5+1 families of R-disjoint sets to limit asdim $\mathbb{H}$ to be less than or equal to five.

The difficulty in finding a cover is due to the different slant the the group has for different A values when multiplying by B. In the other two directions, A and C, this group is very similar to $\mathbb{Z}^3$. Here are two pictures of a ball of radius five centered at (10,0,0) again. The first one is a view in the AB plane and the second is a view in the AC plane. With a little more tipping the second image could line up to look identical to the first, but was slightly slanted to show the peculiarities of balls in the Heisenberg group.


B
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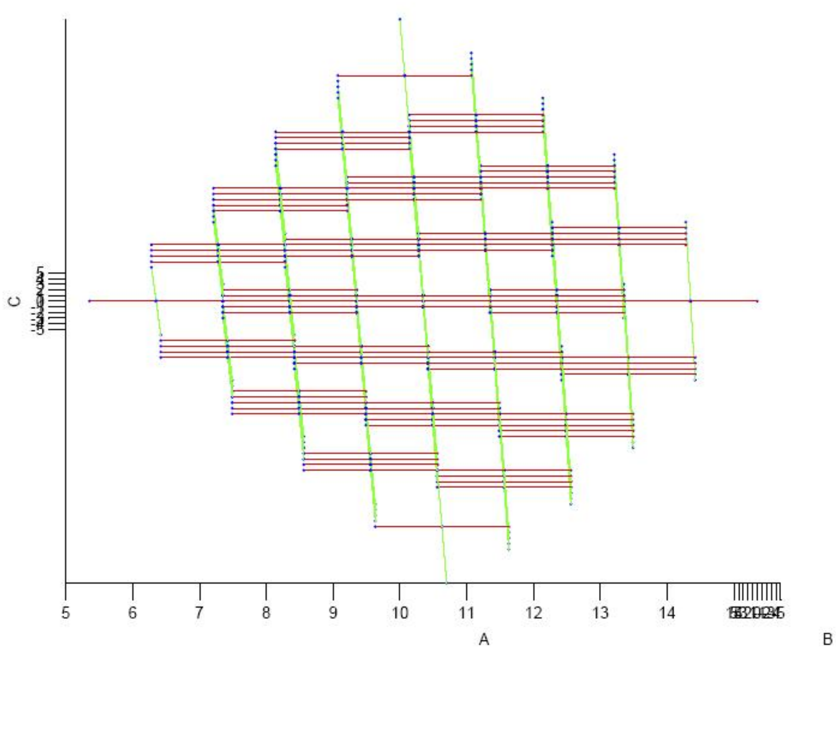
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It is important to note that asdim $\mathbb{H}$ has been found to be three using the following Corollary. This corollary is the conclusion of Carlsson and Goldfarb's paper which is beyond my understanding and the scope of this paper. What makes this corollary unique is it gives an equality about the asymptotic dimension of a group, not an inequality like all of the previous theorems.

Corollary 4.5. *Let* Γ *be a cocompact lattice in a connected Lie group* G*. Let* K *be the maximal compact subgroup in* G*. Then* $\operatorname{asdim}(\Gamma) = \dim(G/K)$*[CG].*

5 Other Findings of the Heisenberg Group

The *semi-direct* product $U \rtimes_\theta V$ is the group with the underlying set $\{(u, v) : u \in U,\ v \in V\}$ and group operation $(u, v) \cdot (u', v') = (u\theta(v)u', vv')$, where $\theta : V \to Aut(U)$ is a group homomorphism[PM]. $\mathbb{H}$ can be represented as the semi-direct product of two subgroups $U = \{(0, y, z) : y,\ z \in \mathbb{Z}\}$ and $V = \{(x, 0, 0) : x \in \mathbb{Z}\}$ with $\theta(x, 0, 0)(0, y, z) = \theta_x(0, y, z) = (0, y, z + xy)$. So for $(x, y, z) \in \mathbb{H}$, $(x, y, z) = \{(0, y, z),(x, 0, 0)\}$which represents the underlying set. Lets look further into the group operation:

$$(x, y, z) \cdot (x', y', z') = \{(0, y, z),(x, 0, 0)\} \cdot \{(0, y', z'),(x', 0, 0)\}$$

$$\begin{aligned}
&= \{(0, y, z)\theta(x, 0, 0)(0, y', z'),(x, 0, 0)(x', 0, 0)\} \\
&= \{(0, y, z)\theta_x(0, y', z'),(x + x', 0, 0)\} \\
&= \{(0, y, z)(0, y', z' + xy'),(x + x', 0, 0)\} \\
&= ((0, y + y', z + z' + xy'),(x + x', 0, 0)\} \\
&= (x + x', y + y', z + z' + xy')
\end{aligned}$$

which is how the group operation was originally defined, and so it is preserved through the semi-direct product. Note that U is a similar group to γ and by identical arguments, U is a normal subgroup of $\mathbb{H}$. Because $\mathbb{H} = U \rtimes V$ we get the following results as a consequences, $U \trianglelefteq \mathbb{H}$, $UV = \mathbb{H}$, and $U \cap V = \{(0, 0, 0\}$. This further understanding of the structure of $\mathbb{H}$ has not yet played a role in discovering the asymptotic dimension, but [DJ] gives the inequality: asymptotic dimension $G_1 \times G_2 \leq$asymptotic dimension G_1+ asymptotic dimension G_2. If this inequality could be adapted to semi-direct products, this would be another way to show that asdim $\mathbb{H} \leq 3$.

Another possibly helpful idea to determine the asymptotic dimension of $\mathbb{H}$ is studying the growth of the cardinality of balls in $\mathbb{H}$. There are two main types of growth of balls that arise in infinite discrete groups, polynomial growth and exponential growth. A group G is said to have *polynomial growth of degree D* if $\frac{1}{C}n^D \leq \#B_n(e) \leq Cn^D$[Du] where $\#B_n(e)$ is the number of elements in the ball of radius n centered at the identity. *Exponential growth* of balls would result in an equation similar to $\#B_n(e) \approx ac^n$. For example, $\mathbb{Z}^n$ has polynomial growth of degree n, and $\mathbb{F}_2$ has exponential growth with base three. I created a computer program that generated Heisenberg balls and counted the number of elements in those balls. Using the data, I was able to come up with a fourth degree polynomial that very accurately modeled the number of elements in larger balls. Rounded to four decimal places, the number of elements in a ball of size n is $0.4306n^4 - 0.1956n^3 + 1.7732n^2 + 1.9711x + 0.9835$. The results for number of elements in each ball was compared to a colleague's and identical results were achieved. So I am confident to conclude that $\mathbb{H}$ has polynomial growth of degree four. This information might prove useful when dealing with balls with an arbitrarily large radius when finding a cover for $\mathbb{H}$.

6 Conclusion

I have acquired a solid understanding of the asymptotic dimension for groups in general. For $\mathbb{Z}^2$ and $\mathbb{F}_2$, an algorithm was created to construct a cover for arbitrarily large D-disjoint sets, which directly proves that their asymptotic dimensions are two and one respectively. Although I have not shown directly by creating a cover for the Heisenberg group that $\operatorname{asdim} \mathbb{H}$ is three, from the theorems and corollary we know that $\operatorname{asdim} \mathbb{H}$ is either two or three. It is greater than or equal to two because it contains a subgroup, γ, that is isomorphic to $\mathbb{Z}$, which has asymptotic dimension of two. It is less than or equal to three as a result of two theorems, one for a homomorphism and the other for Hirsch length. The best cover that I created shows that $\operatorname{asdim} \mathbb{H}$ is less than or equal to five. Because both of the theorems result in three as the bound for asymptotic dimension and because $\mathbb{H}$ is naturally embedded in $\mathbb{Z}^3$, I am lead to believe that the asymptotic dimension of the Heisenberg group is three.

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