Zipper Algorithm

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Broadly speaking this research project focuses on the properties of conformal maps and on how to find a conformal map. A conformal map is a map that preserves the angle between two smooth arcs. For example if two curves in a domain have an angle ω from curve 1 to curve 2, then the image curves under a conformal map will have angle ω between them, where ω is taken as the angle from the image curve of curve 1 to the image curve of curve 2. For more detail consider the following:
Let $f(z) = w$ be analytic at z_0 such that $f(z_0)$ is not zero. Let C be a smooth arc passing through z_0. When:
$z(t) = x(t) + iy(t)$ is a parameterization of C, $w(t) = f[z(t)]$ is a parameterization of the image curve, Γ, under w.
So then $w'(t) = f'(z)z'(t)$ and arg w' = arg f' + arg z'.

Suppose that the tangent line to C at z_0 makes an angle θ_0 with the real line. On the w -plane, the tangent line to Γ at z_0 will have angle $\phi_0 = \psi_0 + \theta_0$. We see the image of C has been rotated by ψ_0, the value of $\arg f'(z_0)$. Now take two smooth arcs, C_1 and C_2in the z-plane such that both C_1 and C_2 pass through z_0. We can again define angles θ_1 and θ_2 such that θ_i corresponds to the angle the tangent line to C_i at z_0 makes with the real line, $i = 1, 2$. Say that under $w = f(z)$, C_1 has image curve Γ_1 and C_2 has image curve Γ_2. Then, by an argument the same as above, the angle of the tangent lines of image curves relative to the real line are:

$\phi_1 = \psi_0 + \theta_1$, for Γ_1 and $\phi_2 = \psi_0 + \phi_2$ for Γ_2. The angle between Γ_1 and Γ_2 is $\phi_1 - \phi_2 = \psi_0 + \theta_1 - \psi_0 + \theta_2 = \theta_1 - \theta_2$

the same as the angle between C_1 and C_2 in the z-plane. So the angle between two curves in the z-plane is the same as the angle between their images in the w-plane.

We say that $w = f(z)$ is conformal at z_0. What we have here is a conformal map, $f(z)$, at z_0. Since $f(z_0)$ is differentiable and $f'(z)$ is not zero then by the inverse function theorem, f has an inverse near z_0. If w is conformal everywhere in the domain and also one to one, w has an inverse everywhere.

Sometimes a problem is presented in such a way that the problem is easiest solved in another domain, for exmaple, due to symmetry in the problem. As we have seen, with a conformal map one may to go from one domain to another and back, allowing an easier solution to a problem easier to solve in another domain. Many boundary problems can be solved using conformal maps by doing just that. Wave propagation is commonly a boundary problem and since much of nature is described in waves, conformal maps have applications in many branches of science. The problem arises in the fact that although a conformal map may be known to exist, for example by application of the Reimann mapping theorem, finding this map is usually no easy task without any general way to be approached. To solve the special case of finding the conformal map that takes the upper half plane to some polygon H.A. Schwarz and E.B. Christoffel independently discovered the function that is now called the Schwarz-Christoffel transformation.

The Schwarz-Christoffel transformation maps the real line in the z-plane onto a the boundry of a simply connected closed polygon in the w-plane. The upper half plane is mapped to the interior of the polygon. The transformation was found in the following manner:

Think of some polygon in the w-plane, where the sides of the polygon never cross each other and the polygon has counter-clockwise orientation. Call the n vertices of this polygon w_k. In order to trace out this polygon the derivative of the mapping function has to be constant on w_{k-1} to w_k and then change to another constant from w_k to w_{k+1}. The function $f(z)$ satisfies this condition when:

$$f'(z) = A(z - x_1)^{-k_1}(z - x2)^{-k_2} \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot (z - x_n)^{-k_n}$$

for x_j the point on the real line in the z-plane such that $f(x_j) = w_j$ and $k_j\pi$ is the exterior angle of w_j.
It must be that

$$\sum k_j\pi = 2\pi, \sum k_j = 2, -1 < k_j < 1$$

Notice that here all of the vertices of the polygon are finite, if one of the vertices are at infinity then $f'(z)$ would end with $(z - x_{n-1})^{-k_{n-1}}$ Clearly, if $f'(z)$ can be integrated, we have found the desired conformal map. Direct integration would yield an undefined function at the bad points x_j. After some massaging of the formula (chapt. 10 [1]) we find that we can integrate $f'(z)$, yielding the definition of the Schwarz-Christoffel transformation, $f(z)$ by:

$$f(z) = A \int (z - x_1)^{-k_1} (z - x2)^{-k_2} \cdot \cdots \cdots \cdots \cdot (z - x_n)^{-k_n} + B$$

It is difficult to have enough time in a semester to cover all of the topics dealing with the theory and applications of complex variable. For this reason, conformal mapping is only briefly discussed in the undergraduate complex analysis course at the U of A. Therefore this research project began with the study of conformal maps and their application. In particular, three conformal maps were studied. The first map,

$$h(z) = \frac{z}{1 - \frac{z}{b}}$$

for $b > 0$ and $b \in \mathbb{R}$,is well defined everywhere except at $z = b$.If the domain, D, of h is restricted to $Imz > 0$ then h is defined everywhere and is analytic everywhere. Also, $h'(z)$ is not equal to zero anywhere on this domain. This shows that h is conformal in D. h maps the extended real line in the z-plane to the extended real line in the h-plane. This can be seen by looking at $b < z$, $z < 0$, $z = 0$, $b > z > 0$, and $z = \infty$:

$$h(b < z) = (-\infty, -b)$$

$$h(z < 0) = (-b, 0)$$

$$h(0) = 0$$

$$h(b > z > 0) = (0, \infty)$$

$$h(-\infty) = h(\infty) = -b$$

This shows that h maps the real line in the z-plane to the real line in the h-plane. To see where the upper half plane is mapped we just need to check where a point in the upper half plane is smapped to since if we consider the extended real line on the z-plane as a circle throught the point at infinity we know the interior of the circle, the upper half plane, will map to the interior of the circle in the h-plane. To do so, consider the point i. Now $h(i)$:

$$h(i) = \frac{i}{1 - \frac{i}{b}}$$

$$= \frac{bi}{b - i}$$

$$= \frac{i}{1 - \frac{i}{b}} \frac{b+1}{b+1}$$

$$= \frac{ib^2 - b}{b+1}$$

Since $b > 0$, $h(i)$ is in the upper half plane, meaning the upper half plane maps to the upper half plane.

The second map, $g(z) = \sqrt{z^2 + c^2}, c \in \mathbb{R}$ can be decomposed to $g(z) = g_2 \circ g_1$ where $g_1(z) = z^2 + c^2$ and $g_2(z) = \sqrt(z)$.Let the domain be the upper half plane minus the segment [0, ic]. It is very easy to see that g_1 maps the real line in the z-plane to (c^2, ∞). Also, a point in the upper half plane will be mapped to some point not on the half line (c^2, ∞). Upon applying g_2, (c^2, ∞) goes to $(-\infty, -c) \cup (c, \infty)$ and the points that were not on (c^2, ∞) go to th upper half plane. We see that removing the segment [0, ic] from the domain is important because this way we don't hit the points on the real line that are the points where g_2 is not well defined. Therefore the mapping g on the domain with [0, ic] removed maps the upper half plane to the upper half plane in a conformal manner.

After the study of these maps I studied the Schwarz-Christoffel transformation. The next thing to do in this project will be to analyze methods of finding conformal maps. One method utilizes the Schwarz-Christoffel transformation. This algorithm has been made into a package to be used in MATLAB, called the Schwarz-Christoffel Toolbox, which can be downloaded for free from Toby Driscoll's software page [4] on the University of Delaware web page. I will be using this program to find how the error of the map and time it takes to find a map change with different inputs. The second method utilizes another transformation, written by another student, Petr Moravsky. The error and time of his algorithm will then be compared with that of the Schwarz-Christoffel algorithm.

The Schwarz-Christoffel toolbox maps according to the map shown above. To do so you must input some set of coordinates to make the polygon on the w-plane. T. Driscol also mdofied the Schwarz-Christoffel map so that the unit disk, the infinite strip $0 \leq Imz \leq 1$, or a rectangle map to the chosen polygon. It is possible to view planes of the domain and codomain simultaneously in the graphical interface in MATLAB. Some pictures of this are shown at the end of this paper. Each picture shows both domains, the codomain being the figure on the left. The first figure shows the unit disk mapped to the funny looking polygon. The black dots are the prevertices. It is dificult to see from the figure, but the two prevertices in the frist quadrant are actually three points; there are two prevertices almost on top of each other. The second figure shows finite prevertices as the black dots of the infinite strip $0 \leq Imz \leq 1$. In the second figure the infinite strip is mapped to the polygon to its left.

After the chosen domain has been mapped to the input polygon points with some level of certainty, information about the map and the error can be found. We can see how the error of the map found and the time it takes to find the map change as a functionof data entered by entering larger or smaller sets of points to describe the same polygon. It is expected that as you add more points to define the polygon the error of the map will decrease but the time it takes to find the map will increase. To find such data on the Toolbox algorithm test cases will be run, that is knowing a confromal map from one domain onto a co-domain, we can see how close to the known domain we get, and how long it takes to calculate such a map. Specifically, we will be looking at $f(z) = \frac{rz}{1+(rz)^2}$, $r \in \mathbb{R}$, from the unit disk. The first thing to do is select points on the unit disk and run them through f. The set $\{f(z)|z \in |z| = 1\}$ is the set of points of the boundary of the image. Call that set B, which has N elements. B can be put in to the Toolbox and then the toolbox will tell us the domain calculated by the Schwarz-Christoffel transformation. Since the true domain is known we can calculate the error in using the toolbox, and also note the time it took the toolbox to find the map. The interesting part comes when we find the error as a function of N and the time as a function of N.

This data of error and time will be compared to similar data from the algorithm written by Petr Moravsky. It may be that one algorithm is better suited for a particular set of points or for a specific purpose, depending on the accuracy intended or the time constraints. Knowing this information could become very useful in applications. Many times the problem encountered in applications is that an algorithm, while being very accurate, is very slow, or the opposite, an algorithm is fast with large error. Being able to specify the case where these two algorithms, the toolbox and Petrs, have strengths and weaknesses would allow someone finding such maps to make an educated and perhaps more efficient choice about the algorithm being used.

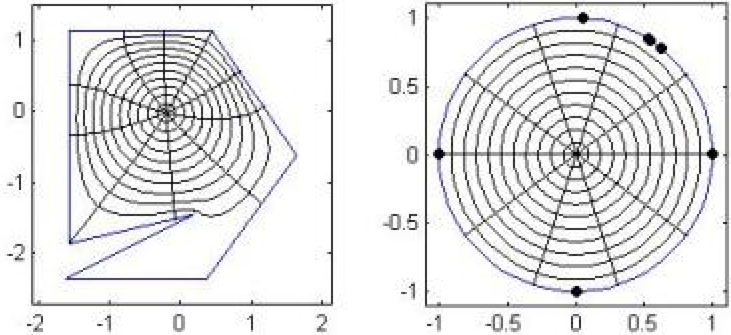
References

[1] Ruel V. Churhill, James W. Brown, Roger F. Verhey *Complex Variavbles and Applications* 1974: McGraw-Hill

[2] Complex Variables *Stephen D. Fischer* 1990: Dover Publications, N.Y.

[3] Donald E. Marshall, Steffen Rohde *Convergence of the Zipper algorithm for conformal mapping* preprint.

[4] Toby Driscoll *Schwarz-Christoffel Toolbox Users Guide*
software link: http://www.math.udel.edu/ driscoll/software/.



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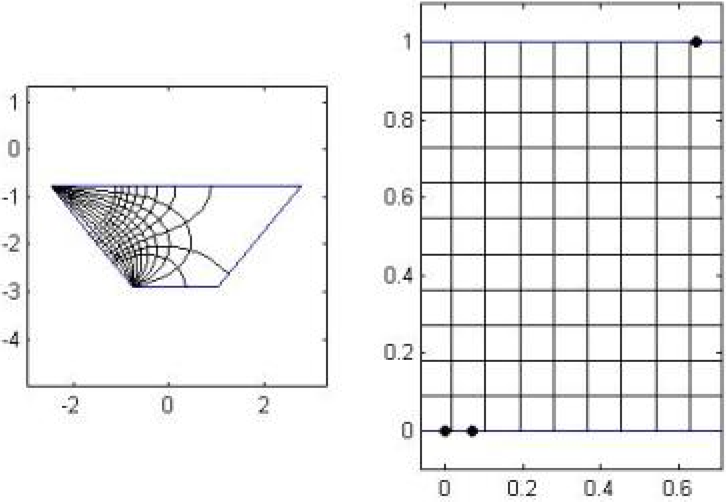
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