Theoretical Models of Oxygen Transport

URA Final Report

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The microcirculation is the body's network of its tiniest blood vessels. Over the last semester I have developed and studied mathematical models of oxygen transport in the microcirculation using the Krogh cylinder arrangement, Figure 1, as a starting point to simplify things. I have studied models of one-dimensional and three-dimensional diffusion to show how far oxygen will diffuse until it stops. It will only diffuse about 47 μm from the capillary into the tissue cylinder given a typical initial partial pressure of 100 $mmHg$. I have also developed graphs showing how diffusion distance changes as the initial partial pressure or oxygen consumption changes.

To develop a model for the axial decline of oxygen as blood flows along a capillary, we must first define the parameters Q, H, and C_0: Q for the blood flow rate, H for the O_2 concentration in red blood cells, and C_0 for the red blood cell concentration in blood. If we use M for oxygen consumption, r_t for the radius of the Krogh tissue cylinder, S for oxygen saturation, and z for distance along the capillary, we can write the following equation

$$S(z + \Delta z)QHC_0 = S(z)QHC_0 - \pi r_t^2 M \Delta z$$

from which we can easily obtain the differential equation

$$\frac{\partial S}{\partial z} = -\frac{\pi r_t^2 M}{QHC_0} \tag{1}$$

Solving this equation is a simple matter of integrating both sides. Doing this, we find that, according to this model, oxygen saturation declines linearly as a function of distance along the capillary. This is true for some of the time. However, this model has a fundamental problem. This model assumes that the amount of oxygen consumed in a particular cross-section of tissue is always given by $\pi r_t^2 M \Delta z$, but this is only the case when the Krogh tissue cylinder is fully oxygenated. At some point, there will always be hypoxic regions, and then the amount of oxygen consumed in a particular cross-section of tissue can more accurately be given by $\pi r_{max}^2 M \Delta z$, so we can now write a more accurate model, which we use when $r_{max} < r_t$

$$\frac{\partial S}{\partial z} = -\frac{\pi r_{max}^2 M}{QHC_0} \tag{2}$$

This equation cannot be solved analytically, but the Euler Algorithm yields a satisfactory solution graph, given that $S \approx 0.9621$ when $z = 0$.

An interesting question we can ask is whether or not oxygen will run out as blood flows along the capillary. The graph of S obtained using numerical analysis implies that S goes to 0 and becomes negative at some point. Further analysis also seems to imply

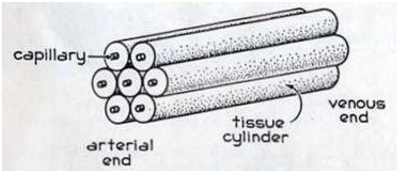

Figure 1: Krogh cylinder arrangement. Source: "The Krogh Cylinder"

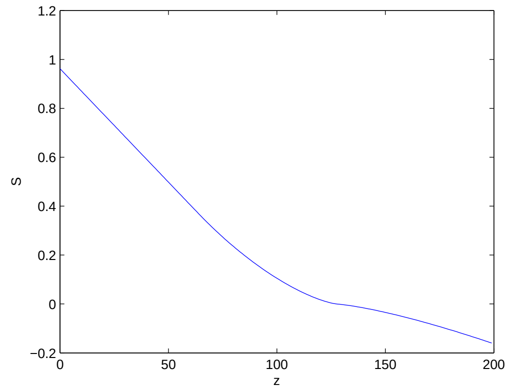

Figure 2: Axial decline in oxygen saturation

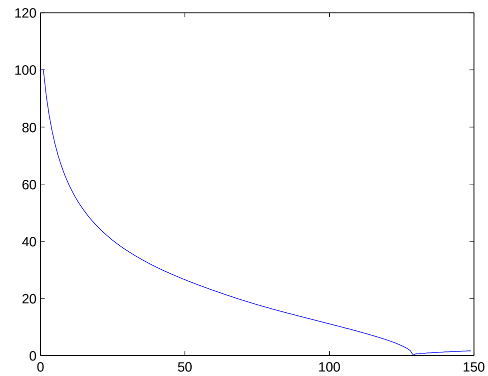

Figure 3: Change in partial pressure as a function of distance along capillary

this behavior. Graphs of P, the partial pressure of oxygen, and $\frac{\partial S}{\partial z}$ seem to show that oxygen saturation will go to zero as blood flows along the capillary.

However, all of these graphs are only approximations of the actual behavior. We can use them to estimate whether or not oxygen will run out, but we cannot prove anything this way. For a proof, we will have to use other methods. The equation for S in terms of P is

$$S = \frac{P^n}{P^n + P_{50}^n}$$

where $n \approx 2.4$ and $P_{50} = 26$. If $P \ll P_{50}$, then we can approximate S as

$$S = \left(\frac{P}{P_{50}}\right)^n$$

If we define w as the distance that oxygen will diffuse using a one-dimensional model, we can approximate $\frac{\partial S}{\partial z}$ as

$$QHC_0 \frac{\partial S}{\partial z} = 2\pi r_c w$$

The equation for w is $w = (\frac{2D\alpha P}{M})^{\frac{1}{2}}$, so we can approximate the rate of consumption per unit length by $2\pi r_c(\frac{2D\alpha P}{M})^{\frac{1}{2}}$ for small values of P. Now, factoring in the approximation of S for small values of P we arrive at the following differential equation

$$QHC_0 \frac{\partial}{\partial z} \left(\frac{P}{P_{50}} \right)^n = -2\pi r_c \left(\frac{2D\alpha P}{M} \right)^{\frac{1}{2}} \tag{3}$$

which we can easily solve by separation of variables. The solution is

$$P = \left[-\frac{(n-1/2)(2\pi r_c (\frac{2D\alpha}{M})^{\frac{1}{2}} P_{50}^n z}{QHC_0 n} + c \right]^{\frac{1}{n+1/2}} \tag{4}$$

This equation gives us a far more definitive answer to the question of whether or not oxygen will run out as it flows along the capillary. Clearly, in this equation, P goes to zero as z increases. If P goes to zero, then S goes to zero, and at some point, the oxygen saturation will be zero. So, as blood flows along the capillary, it will eventually run out of oxygen. Figure 2, which was generated using the Euler algorithm, gives an approximation of when this will happen.

We can also develop a model for oxygen saturation along the alveolar-capillary barrier in the lung that is similar to the previous model. The alveolar-capillary barrier is a thin blood gas barrier between the alveoli and capillaries in the lung. It is only about 600-800 nm thick. In this region, oxygen saturation will be an increasing function of distance along the capillary.

In order to write this model, we will need to define a few new parameters. P_a will denote the partial pressure of oxygen in the alveoli, which is about 140 $mmHg$; P_b will be the partial pressure of oxygen in the capillary, which changes with distance along the capillary; h will be the thickness of the capillary wall, about 2 μm; and w will be the depth of capillary wall across which oxygen is diffusing, which we will say is about 5 μm. With this information, we find that the amount of oxygen gained in a particular cross-section of tissue is given by $D\alpha(\frac{P_a - P_b}{h})w\Delta z$, and we have an equation similar to the equation for axial decline of oxygen

$$S(z + \Delta z)QHC_0 = S(z)QHC_0 + D\alpha \left(\frac{P_a - P_b}{h} \right) w\Delta z$$

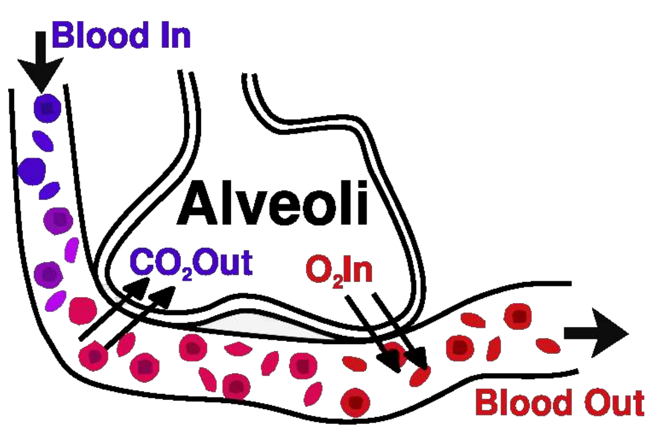

Figure 4: Gas exchange at the alveolar-capillary boundary. Source: Wikipedia

Rearranging this equation, we arrive at the following differential equation for the change in oxygen saturation along the alveolar-capillary barrier

$$\frac{\partial S}{\partial z} = \frac{D\alpha(\frac{P_a - P_b}{h})w}{QHC_0} \tag{5}$$

Since P_b is dependent on S, this equation cannot be solved analytically, so we must solve it numerically, using a method such as Euler's algorithm. With further research, a proof of whether oxygen saturation at the alveolar-capillary boundary crosses $S = 1$ or assymptotes at $S = 1$ could be found.

Over the last semester and a half, I have shown that oxygen transport in the microcirculation occurs with short distances. I have explored how oxygen diffuses radially depending on oxygen consumption or initial partial pressure. I have also shown how oxygen saturation declines with distance along the capillary and how it increases at the alveolar-capillary boundary. These models are never entirely accurate. Assumptions are always made in mathematical models, but they can usually give an accurate enough model to approximate how real-life phenomena will occur.

References

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