Estimating VaR using Copula.

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Introduction:

A copula is a function which relates a multivariate distributional function to a lower dimensional marginal distributional function, generally a one-dimensional function. The concept of copulas was introduced by Sklar in 1959. But recently it has become a very important statistics tool having applications in fields like actuarial science, finance, biological modeling and engineering. To get a good understanding of copulas one needs to get familiar with some statistical concepts. But to study its applications, knowledge about the field in which it's being applied is also relevant. In this paper we will study the concept of copula and its application. Although copula has its application in many fields, but its applications in the financial fields have become extremely popular these days.

We will be looking at its application for estimating Value at Risk (VaR), but estimating VaR for portfolio requires good understanding of some financial tools. One such financial tool used for estimating VaR is "Credit Metrics", so we will be discussing the working of Credit Metrics and how to use copula along it for estimating VaR. Once we are in position of discussing its practical applications we will use "mlCopulaSelection", "Copula" and "CreditMetics" package in R Project software to see its applications. We will be dividing this paper in three sections, in the first section we will discuss some statistical concepts important to understand copula and then finally introduce the concept of copula. In second section we will be discussing Credit Metrics and how to use copulas for calculating VaR. Finally in the last section we construct as small portfolio of bond and estimate its VaR using R software.

Section 1

Statistical concepts and copula functions :

Joint Distribution Function:
Consider a function f(x,y),

$$\mathbf{f(x,y) \geq 0} \qquad -\infty<\mathbf{x}<\infty,\ -\infty<\mathbf{y}<\infty$$

&

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$$

If a function f(x,y) satisfies these conditions then it is a joint distribution function,

Cumulative density function:
From the definition of joint distribution function, consider a function F(b,d) such that

$$\mathbf{F(b,d)} = \mathbf{Pr(x \leq b\ \&\ y \leq d)} \qquad -\infty<\mathbf{x}<\infty,\ -\infty<\mathbf{y}<\infty$$

$$= \mathbf{Pr(-\infty \leq x \leq b\ \&\ -\infty \leq y \leq d)}$$

$$= \int_{-\infty}^{d}\int_{-\infty}^{b} f(x,y)\,dxdy$$

Then F(b,d) is a Joint Cumulative Density function.
In general, F(b,d) gives the probability such that x ≤ b and y ≤ d. Which is given by the volume under the surface F(b,d) in the region -∞<x<b, -∞<y<d.

Marginal Distribution Function:

$F_x(x)$ and $F_y(y)$ are Marginal Distribution Function of F(x,y) when,

F(x, ∞) = Pr(X≤x,Y≤∞) Pr(Y≤∞) = 1
F(x, ∞) = Pr(X≤x) = $F_x(x)$ = G(x)
&
F(∞,y) = Pr(X≤∞,Y≤y) Pr(X≤∞) = 1
F(∞,y) = Pr(Y≤y) = $F_y(y)$ = H(y)

So G(x) and H(y) are Marginal Distribution Function of F(x,y)

Copula function:

- Two dimensional copula :
  Consider a joint distribution function

  H(x,y) with domain $[-\infty,\infty]x[-\infty,\infty]$ & range[0,1] $\in$ **I**

  With marginal function,

  $H(x,\infty) = F(x) = u$ &

  $H(\infty,y) = G(y) = v$

  then range of F(x) and G(y) $\in$ **I**

  Consider a function C(u,v) such that,

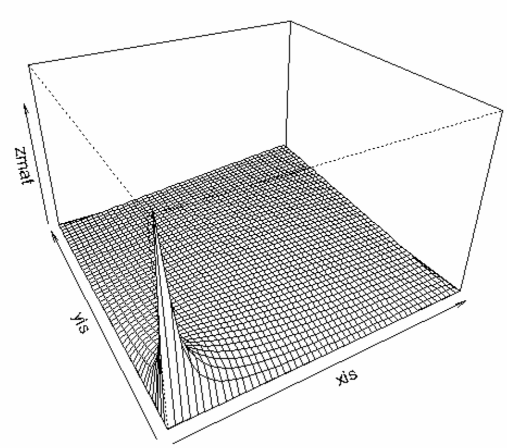
  $C(u,v) = C[F(x),G(y)] = H(x,y)$

  then C(u,v) is called a copula function if it satisfies the following properties.

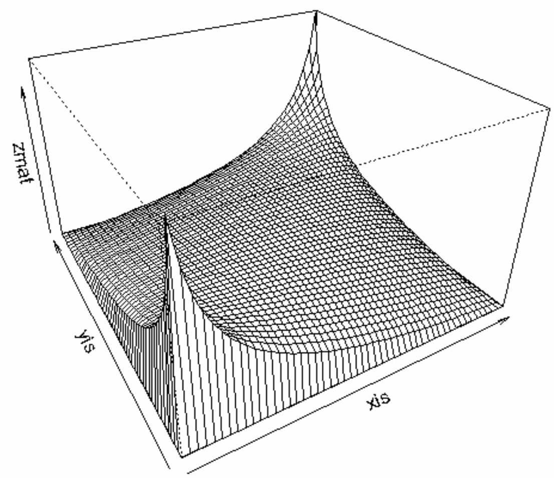
  1) DomC(u,v) = [0,1]x[0,1] $\in \mathbf{I_2}$ &
     RanC(u,v) = [0,1] $\in$ **I**
  2) C(u,v) is a Ground function
     i.e $C(u,0) = 0 = C(0,v)$
  3) C(u,v) is a 2-increasing function
     i.e $C(u,1) = u$ & $C(1,v) = v$
  4) For every u_1, u_2, v_1, v_2 in **I** such that $u_1 \leq u_2$ & $v_1 \leq v_2$ function
     C(u,v) satisfies the inequality,
     $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u1,v1) \leq 0$

  Then function C(u,v) defined as a copula function which relates the Marginal Distribution Function F(x) and G(y) of H(x,y) with function H(x,y) itself. There are many predefined copula function which are commonly used like Archimedean copula, elliptical copula, normal copula and many others. Below are some examples of these copula surfaces corresponding to the density distribution.

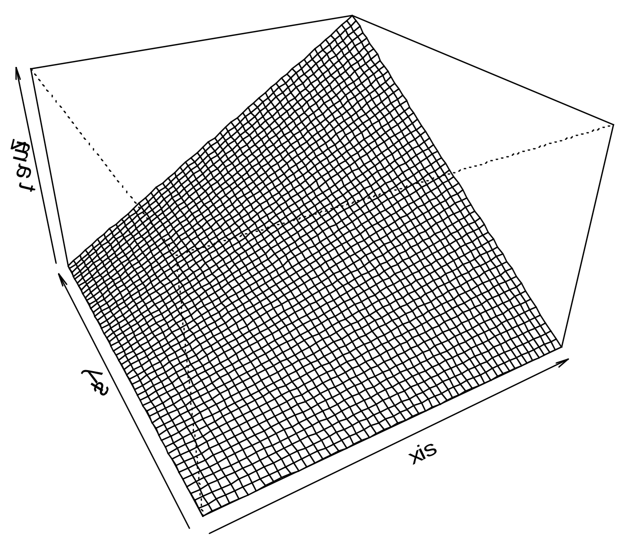
  - Archimedean Copula surface having strong density near the origin.



- Elliptical copula surface have high density distributed at origin and (1,1).



- Normal Copula function is the simplest of all copula function i.e C(u,v) = uv. With zero density near the ‘x’ and ‘y’ axis but increase as it goes away from the axis.



- <u>Multi-dimensional copula:</u>
  Multi-dimensional copula function can be defined in a similarly way as a 2-dimensional copula function.

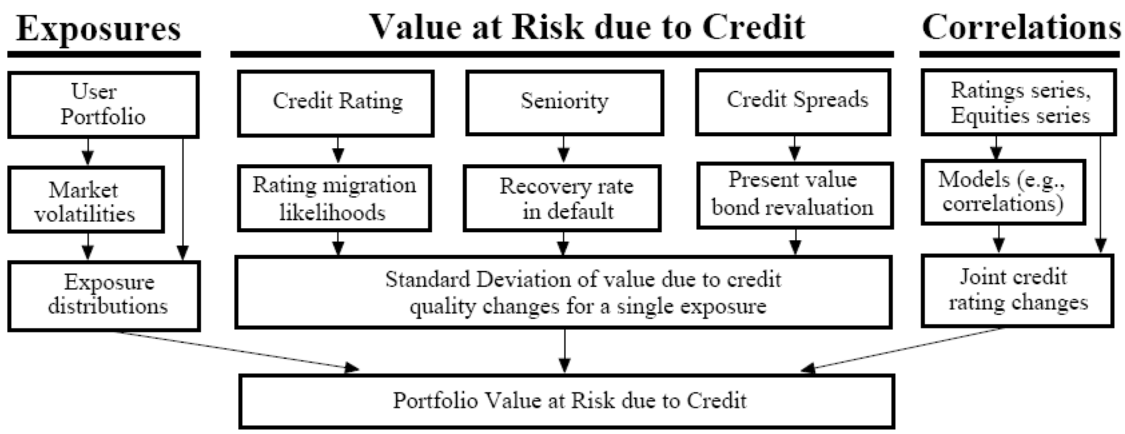
  Consider a joint distribution function $F(x_1, \ldots, x_n)$ for the random variables $X_1, \ldots, X_n$ having marginal functions $F_1(x_1), \ldots, F_n(x_n)$
  and $F(x_1, \ldots, x_n) = C[F_1(x_1), \ldots, F_n(x_n)]$

  Then function $C(u_1, \ldots, u_n)$ relating function $F(x_1, \ldots, x_n)$ with its marginal functions $F_1(x_1), \ldots, F_n(x_n)$ is called a *copula*.

Section 2

Credit Metrics:

Credit Metrics is a tool for assessing portfolio risk and is used widely to find Value at Risk (VaR) of a portfolio. We will be using copula function in Credit Metric to calculate VaR. Credit Metrics estimates risk of portfolio based on the changes in obligators credit quality. But in order to understand the application of copula function in Credit Metrics we need to go through the basics model of Credit Metrics which is used to calculate VaR of a portfolio. Below if a financial model of Credit Metrics:-



This model is based on the technique of migration analysis. This model is divided into 3 categories which contribute for estimating VaR of the portfolio. Exposure and Value at risk due to credit are not very important in terms of implication of copula function. Copula will be introduced in the correlation portion of Credit Metrics. We will be using copula function for estimating the correlation matrix for our portfolio.

Exposure:

Credit Metrics has its applications in may different financial field. It is not limited to estimating bonds value at risk, but has application in areas such as:-

- non-interest bearing receivables (trade credit);
- bonds and loans;
- commitments to lend;
- financial letters of credit; and
- market-driven instruments (swaps, forwards, etc.)

So the Exposures section of the Credit Metrics model depends on the area in which Credit Metrics is being used. In this paper our principal objective is to estimate VaR for a portfolio containing bonds therefore the Exposures section pertains to the bond portfolio. It incorporates the information about the each bond in the portfolio such as bond rating, maturity period, coupon value and seniority of the bond.

Value at Risk due to Credit:

This section is subdivided into 3 sections which contribute in calculating the standard deviation of bond due to credit quality migration, which is the risk associated with the bond.

Credit rating: This portion contains the probability of credit quality migration likelihood based upon the initial rating of the bond. To get these probabilities we use one year transition matrix for bond migration likelihood. To understand it better let us consider an example of bond issued by Toyota Mtr Crop.

Issue	Price	Coupon	Maturity	Seniority	S&P Rating
Toyota Mtr Cr Corp	104.11	5.45	4	Senior Secured	AAA

Therefore probability of credit quality migration for this AAA bond will be:-

Year-end rating	Probability of state (%)
AAA	96.27
AA	2.48
A	0.16
BBB	0
BB	0
B	0
CCC	0.08
Default	0

Seniority: This section deals with the seniority class of the bond, which is used in the case of default. In the case of default the bond value is determined by the probability of recovery for that bond, and value at default will be bond value time's probability of recovery for that bond based upon the seniority of the bond. Base upon the historical data the different senority class and there recovery rate calculated by S&P are as follows:-

Seniority Class	Mean (%)
Senior Secured	53.80
Senior Unsecured	51.13
Senior Subordinated	38.52
Subordinated	32.74
Junior Subordinated	17.09

Credit spreads: Due to different future rating state we need to find the present value of the bond at different rating state. So based on the credit spread for each bond and with the help forward zero curve which is already given to us, we discount the future value of the bond for each future rating state. These forward curves are stated as of the risk horizon and go to the maturity of the bond. Let's calculate the present value of the Toyota bond based on this forward zero curve table:-

Example one-year forward zero curves by credit rating category (%)

Category	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

The present value or the remaining cash flow at the end of the year corresponding to AAA bond maturing 4 year from now will be:

$$V = 5.45 + \frac{5.45}{(1+3.06\%)} + \frac{5.45}{(1+4.17\%)^2} + \frac{109.56}{(1+4.73\%)^3} = \$111.11$$

At the end of first three year cash flow is \$5.45 and at the end of forth year cash flow is 5.45 + 104.11 = \$109.56, so based on the forward zero curve the discounted value at the end of this year for this bond corresponding for AAA rating is \$111.11. With the help of this table we can calculate forward values corresponding to all rating state and value corresponding to default will be bond value times recovery rate at default based on the seniority of the bond, forward value table for Toyota bond will be :

Year-end Rating	Forward Value
AAA	111.1089
AA	110.9651
A	110.5441
BBB	109.6291
BB	105.4496
B	102.2474
CCC	88.19815
Default	56.01118

Credit risk estimate: As we mentioned earlier the standard deviation of bond due to credit quality migration is a measure of the risk associated with that bond. So based on the credit rating, seniority and credit spread we can calculate the standard deviation for the bond. The standard deviation calculation for Toyota bond will be:-

Year-end rating	Probability of state (%)	Forward Value ($)	Probability weighted values (%)	Difference of value from mean ($)	Probability weighted difference squared
AAA	96.27	111.1089303	106.9645672	1.144999509	1.262122686
AA	2.48	110.9651025	2.751934543	1.001171707	0.024858151
A	0.16	110.5440812	0.17687053	0.580150344	0.000538519
BBB	0	109.6291197	0	-0.334811154	0
BB	0	105.4495985	0	-4.514332315	0
B	0	102.2473711	0	-7.716559769	0
CCC	0.08	88.19814936	0.070558519	-21.76578147	0.378999395
Default	0	56.01118	0	-53.95275083	0
		Mean =	109.9639308	Variance	1.66651875
				St Dev	1.29093716

Correlation:

Till now we have been dealing with single bond, but in order to find the risk associated with the portfolio we need to consider joint probabilities of the bonds. Since we know that due to some macroeconomics elements, all bonds are related in some way or other. Since bonds are not independent, we can not multiply likelihood of to bonds to find distribution for whole portfolio. So we need to find the relation between the bonds. There are many different ways of finding correlation. In the technical document of Credit Metrics they discuss 4 different ways of finding correlation. But here we will introduce copula function to find the correlation between the bonds and not use any of the methods suggested in the technical document of Credit Metrics.

We will treat the probability of credit quality migration of a bond in a portfolio as a marginal probability of the portfolio. Suppose we have a portfolio of 2 bonds then credit migration likelihood for two bonds will be two different marginal probabilities for the portfolio. Let these two marginal be given by the vectors U and V. By using R software and "mlCopulaSelection" package in it we can use these marginal to find a copula copula model C(U,V) fitting these marginal probability. Once we have found the copula function we can calculate "Spearman correlation" as :-

$$\text{Rho} = 12\int_0^1\int_0^1 (C(u,c) - uv)dudv = 12\int_0^1\int_0^1 C(u,c)dudv - 3$$

Using different copula function for different pair of bond we can construct a correlation matrix for the portfolio. This correlation matrix along with exposure and VaR due to credit for a bond can be used to calculate the final value at risk for the portfolio.

There are both some advantages and some disadvantages associated with using this method for calculating correlation. Advantage is that we do not need extra information such as the issuer asset value history and other information to calculate correlation, we can calculated it based on the credit migration likelihood provided by transition matrix.
The disadvantage is that correlation is found on the bases of credit rating and not between the bonds it self, which means that it will not consider the other diversification effects such as bonds from different issuers are different even if they have same rating.

Section 3

Application:

In this section of the paper we will be applying Credit Metrics and copula on a 4 bond portfolio. For simplicity we have chosen all bond such that they are senior secured, which implies that recover rate for all 4 of these bonds will be 53.8%. The portfolio consists of the following bonds.

Bond	Issue	Price	Coupon	Maturit	S&P Rating
1	Toyota Mtr Cr Corp	104.11	5.45%	4	AAA
2	Bank New York	106.75	6.38%	5	AA
3	Disney Walt Co	104.7	5.70%	4	A
4	Kroger Co	107.54	6.75%	5	BBB

Transition matrix obtained from S&P's for 2006 is

Table 1 One-Year Transition Matrix (%) (1986-2006)

Rating	AAA	AA	A	BBB	BB	B	CCC/C	D	N.R.
AAA	96.27	2.48	0.16	0	0	0	0.08	0	1.01
AA	0.48	91.99	1.44	0.04	0	0	0	0	6.05
A	0.02	2.01	88.77	0.95	0.04	0.02	0.01	0	8.19
BBB	0	0.05	2.86	87.64	0.64	0.12	0.07	0.01	8.62
BB	0	0.09	0.26	7.88	77.66	3.38	0.69	0.26	9.78
B	0	0	0.31	3.08	6.46	69.54	4.92	1.54	14.15
CCC/C	0	0	0	0	2	9.33	64.67	14	10

Data source: Standard & Poor's U.S. Public Finance ratings.

As described earlier we will be calculating present value or the remaining cash flow at the end of the year for all these 4 bonds for all rating state, these values are given in the table below:

Remaining Cash Flow at Year End

	Bond 1	Bond 2	Bond 3	Bond 4
AAA	111.11	116.62	112.56	118.96
AA	110.97	116.43	112.42	118.76
A	110.54	115.86	111.99	118.19
BBB	109.63	114.68	111.07	116.99
BB	105.45	108.78	106.85	111.01
B	102.25	104.60	103.62	106.77
CCC	88.20	89.17	89.42	91.09
Default	56.01	57.43	56.33	57.86

By using "CreditMetric" in R project and with the above data we can calculate VaR for your portfolio at particular confidence level. But to use Credit Metrics in R we need to know the correlation matrix. As described earlier the correlation matrix can be found by first finding copula function by using bond migration likelihood as margins and the then integrating that function between 0 and 1. In the four bond case we need to find six copula functions to make a correlation matrix.

Correlation matrix:-

We will have four marginals for your portfolio. By taking pair of 2 marginal at a time and using "mlCopulaSelection" in R we can find your models.

R- Code

Defining Marginal
u1<- c(96.27,2.48,0.16,0,0,0,0.08,0)/100
u2<- c(0.48,91.99,1.44,0.04,0,0,0,0)/100
u3<- c(0.02,2.01,88.77,0.95,0.04,0.02,0.01,0)/100
u4<- c(0,0.05,2.86,87.64,0.64,0.12,0.07,0.01)/100

Finding copula model corresponding to these marginal using function in "mlCopulaSelection" package.
res.12<-mlcbbsel(u1,u2)
res.13<-mlcbbsel(u1,u3)
res.14<-mlcbbsel(u1,u4)
res.23<-mlcbbsel(u2,u3)
res.24<-mlcbbsel(u2,u4)
res.34<-mlcbbsel(u3,u4)

Displaying copula model and parameter corresponding to these marginal
res.12$copmax
res.12$parmax
res.13$copmax
res.13$parmax
res.14$copmax
res.14$parmax
res.23$copmax
res.23$parmax
res.24$copmax
res.24$parmax
res.34$copmax
res.34$parmax

Output will be the code for the copula function, along with their constant parameters.
In this case output from R was :-

Marginal of Bond	Model	Rotation	theta	delta
1 & 2	CBB10	180	0.879	0.005
1 & 3	CBB10	0	0.768	0.005
1 & 4	CBB10	180	0.745	0.012
2 & 3	CBB10	180	0.915	0.005
2 & 4	CBB10	180	0.756	0.005
3 & 4	CBB10	0	0.9	0.005

Function corresponding to the copula model CBB10 (theta, delta, u, v) with ‘u’ & ‘v’ as the marginal and ‘theta’ & ‘delta’ as constants.

```
{
S = 1-u^(1/delta);
T = 1-v^(1/delta);
W = theta*S*T;
C = u*v*(1-W)^(-delta);
DuS = (-1/delta)*u^(1/delta-1);
DuW = theta*T*DuS;
DvT = (-1/delta)*v^(1/delta-1);
DvW = theta*S*DvT;
DvuW = theta*DuS*DvT;
C(u,v)          =          (1-W)^(-delta)+v*(-delta)*(1-W)^(-delta-1)*(-
1)*DvW+u*delta*(1-W)^(-delta-1)*DuW+u*v*delta*(-delta-1)*(1-W)^(-delta-
2)*(-1)*DvW*DuW+u*v*delta*(1-W)^(-delta-1)*DvuW
}
```

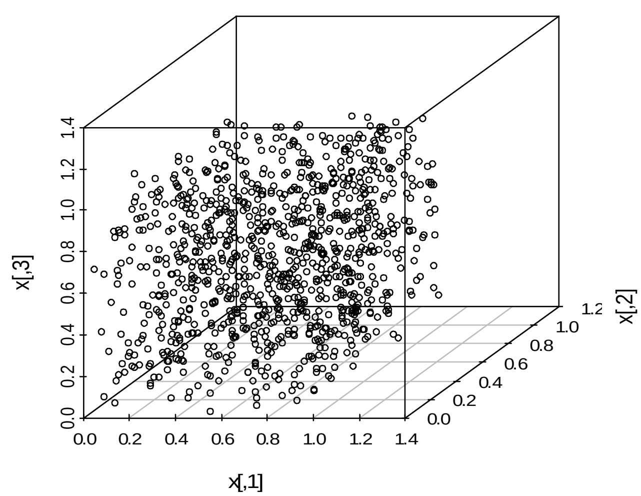
With the help of R we can construct a density distribution for this copula model. Code for graphing its density distribution is as follows:

will generate 1000 random variable
u<- runif(1000)
v<- runif(1000)

will produce a density distribution for u and v
w<- dcbb10(.005,.879,u,v)

will convert u,v,z in form of matrix and generate its scatter plot
x<- matrix(c(u,v,w),1000,3, byrow = TRUE)
scatterplot3d(x)

Density distribution for CBB10 :



By using this copula model we can find Spearman correlation by performing a double integration on this function from 0 to 1.

$$\text{Rho(1,2)} = 12\int_0^1\int_0^1 C(u,c)\,du\,dv - 3$$

$$\text{Rho(1,2)} = 12\int_0^1\int_0^1 \frac{1}{((1-.8795u+u^{200})+(1+v^{200}))^{.005}}\,du\,dv \ - 3$$

Rho(1,2) = .75

Similarly, we can calculate rho for all six copula function and construct a correlation matrix for this portfolio. Correlation matrix for your 4 bond portfolio is:--

Rho	Bond 1	Bond 2	Bond 3	Bond 4
Bond 1	1	0.75	0.299	-0.469
Bond 2	0.75	1	0.76	-0.012
Bond 3	0.299	0.76	1	0.574
Bond 4	-0.469	-0.012	0.574	1

Now with all of the above data we can calculate VaR of your portfolio using "CreditMetrics" package in R project.

R-Code for "CreditMetrics"

```
        ## Number of Bonds in portfolio
N <- 4
        ## number of simulations
n <- 100000
        ## Fixed interest rate based on Treasury Bill
r <- 0.025
        ## Exposure at risk or the face value of the bond.
ead <- c(104.11,106.75,104.7,107.54)
        ## Rating category.
rc <- c("AAA", "AA", "A", "BBB", "BB", "B", "CCC", "D")
        ## Loss at default = 1- recovery rate for your bond
lgd <- 0.462
        ## Rating of Bonds in portfolio
rating <- c("AAA", "AA", "A","BBB")
bondnames <- c("Bond 1", "Bond 2", "Bond 3","Bond 4")
        ## confidence level at which VaR is being calculated
alpha <- 0.99

        ## correlation matrix for your portfolio
rho <- matrix(c(      1,     .75,   0.299, -0.469,
                     0.75,       1,    0.76, -0.012,
                    0.299,    0.76,       1,   .574,
                   -0.469, -0.012,   0.574,       1),4,4,dimnames =
list(bondnames,bondnames), byrow = TRUE)
```

```
        ## one year transition matrix form standard&poors website

M <- matrix(c( 90.81,  8.33,  0.68,  0.06,  0.08,  0.02,  0.01,   0.01,
                0.70, 90.65,  7.79,  0.64,  0.06,  0.13,  0.02,   0.01,
                0.09,  2.27, 91.05,  5.52,  0.74,  0.26,  0.01,   0.06,
                0.02,  0.33,  5.95, 85.93,  5.30,  1.17,  1.12,   0.18,
                0.03,  0.14,  0.67,  7.73, 80.53,  8.84,  1.00,   1.06,
                0.01,  0.11,  0.24,  0.43,  6.48, 83.46,  4.07,   5.20,
                0.21,     0,  0.22,  1.30,  2.38, 11.24, 64.86,  19.79,
                   0,     0,     0,     0,     0,     0,     0,    100
                )/100, 8, 8, dimnames = list(rc, rc), byrow = TRUE)

        ## function returning VaR

cm.CVaR(M, lgd, ead, N, n, r, rho, alpha, rating)
```

Using these codes in R we were able to compute the VaR for your portfolio, and output from R was VaR at 1% confidence was $9.5. This means that based on the reliability of this method we are 99% sure that at any given time portfolio will not have loss any more that 9.5$. So based on this method VaR for your 4 bond portfolio is $9.5 at 1% confidence.

Problems in the model:
There are few problems with this approach of calculating VaR. Firstly as discussed earlier your model if based on correlation between the bond ratings, so it will not account for the diversification of the nature of the bond. Which means a bond issued by Toyota Mtr Inc with bond rating AAA will be treated at par with bond issued by Bank of New York with AAA rating. But since the process of bond rating used by S&P and other rating agencies are so detailed and account all aspects of the companies, so the effect of diversification is minimized and we can say that bond rating also account for diversification to certain extent. Second problem with this model is that "mlSelectionCopula" package used to find copula model contains only 40 models, therefore the choice for the copula model if limited to these 40 models. So the correlation matrix obtained by this process might not be that accurate. In future these problems can be over come and this model can be improved.

Conclusion: In this paper we looked at the basic properties of copula function and some copula functions. We also looked at the working of Credit Metric and introduced the copula function to it in order to estimate VaR of a portfolio. Finally we looked at an example of a 4 bond portfolio and calculated VaR of this portfolio using Credit Metric and copula function. Our final result for this 4 bond portfolio is VaR at 1% confidence is $9.5 for this portfolio.

Sources:

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“CreditMetrics” Package URL : *http://cran.r-project.org/doc/packages/CreditMetrics.pdf*