Research on Student Learning of College Algebra, Pre-Calculus and Calculus Concepts

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Introduction

The purpose of this project is to create a resource that could help college mathematics instructors gain a better understanding of the most common problems their students have in relation to the content of their courses. Specifically this project focuses on the content of College Algebra, Pre-Calculus and Calculus. The primary sources collected are published research studies relating to the teaching and learning of these mathematics classes. Classroom lessons that address the problems in the research studies are provided as well. These lessons are to give concrete examples and ideas for how instructors can specifically address the problems they are most likely to encounter in their classrooms.

Summaries of the Research Studies

Schwingendorf, K., McCabe, G., & Kuhn, J. (2000). A longitudinal study of the C^4L calculus reform program: Comparisons of C^4L and traditional students. *Research in Collegiate Mathematics Education, 4*(8), 63-76.

Research Questions:

- Which program, C^4L (Calculus, Concepts, Computers and Cooperative Learning) or TRAD (Traditional way), provides a student with better understanding of the required calculus concepts?
- Which program, C^4L or TRAD, better inspires students to pursue further study in calculus or more generally, mathematics?

Method:

A total of 4,634 students enrolled in either the C^4L or TRAD program were used in this study. The following three statistics were used in comparing the student's level of understanding calculus concepts: average final grade, average final grade for all mathematics courses taken after completing the calculus programs, and last available overall grade point average. The following two statistics were considered when comparing student's level of interest in calculus or mathematics courses: average number of calculus courses and average number of non-calculus mathematics courses. Additionally, there appeared to be potentially influential extraneous factors that would affect the results. In light of this the following three variables were also taken into account: predicted grade point average, major course of study and gender. A statistical model was then used taking into account all of the criteria listed above to determine the effectiveness of these two types of classrooms. The estimated difference between the two programs was computed using this statistical model. The unadjusted results excluded the extraneous variables while the adjusted results included them.

Findings:

Both the adjusted and unadjusted results indicated the same outcome. The average final grades for the students in the C^4L program were almost half a grade higher than that of the TRAD students. In relation to the average grades of either group in the mathematical courses beyond calculus there appeared to be no difference. Meanwhile there appeared to be only the slightest difference in favor of the C^4L program regarding the overall grade point average. On average the C^4L students took more calculus courses following the completion of the C^4L program. It appeared however that the average number of non-calculus mathematics courses taken beyond the C^4L program was the same as that of those who had taken the TRAD program.

Implications:

The implications of this research are that students learn better in situations where they are not being fed the information almost exclusively through lecture format as was occurring in the TRAD program. The C^4L program required the students to participate which produced a more thorough understanding of the content as demonstrated by the students' success. The research shows a higher content understanding and retention with a greater enthusiasm for future calculus courses occurs when the students are actively involved in the learning through working with their peers and doing group computer activities to evaluate mathematical concepts.

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education, 4*(8)*,* 103-127.

Research Questions:

- How is a particular students understanding of the concept of derivative similar to and different from the concept as seen by the mathematical community?
- How are individual student understandings of the concept similar or different?
- Do students tend to learn the various aspects of the concept in a particular developmental order? Are some aspects of the concept usually learned before others?

Method:

This study followed nine students in an Advanced Placement high school calculus course. The data collected in this particular research project is from a series of five interviews that spread across the nine months of the calculus course. The students were asked to answer open-ended questions and solve problems involving the concept of derivative. The data collected was organized in a framework that considered two things. The first thing considered was the way a derivative has multiple representations because it can be represented either graphically as a slope, verbally as a rate, physically as a velocity or symbolically as the difference quotient. The second thing is that the derivative involves the ratio process, the limiting process and the derivative function each of which can be paired with the aforementioned four representations.

Findings:

The findings reveal that these students have surprising weaknesses in their understanding of the concept of derivative. While the concept of derivative may have a rich variety of meanings and uses for those in the mathematical community many students have only a surface level understanding of this concept. Additionally, the order in which the students build their knowledge varies. However as students begin to piece different aspects together their understanding becomes more aligned. The findings also reveal that there is no sort of hierarchy by which the students learn the diverse aspects of the concept of derivative.

Implications:

The research in this study reveals that although students may appear to have mastered the concept of derivative this is not always the case as there were many holes in the students' understanding. Since the students in this study are considered to be some of the best in the United States it is safe to assume that if they do not have a complete and solid understanding of the concept of derivative then the rest of the student population taking calculus courses probably does not either. This study suggests that the challenge for educators is to give students experiences that will enable them to understand the concept of derivative in all aspects as discussed in this article. Finally another important aspect that this research implies is that one must consider if the textbook or curriculum materials are intentional to provide opportunities for students to learn these processes.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Research Question:

- What type of reasoning do students employ regarding covarying quantities in dynamic situations?

Method:

The twenty students who participated in this study had all received a course grade of A in their recently completed 2nd-semester calculus classes. These students were asked to analyze and then record their observations in written form of five different dynamic situations. Their answers were then analyzed according to their understanding of the two varying quantities and the ways these quantities change in relation to each other. Six of these students were additionally selected to participate in 90-minute interviews to further expound on their reasoning.

Findings:

The findings revealed that overall the students understood the relationship between dependent and independent variables in dynamic situations. Additionally the students demonstrated an awareness of how the changes in the input of a function affected the amount of change in the output. The students were inconsistent however when considering uniform increments of the input in relation to understanding the rate of change of the output. Overall the students demonstrated that they were unable to coordinate the instantaneous rate of change with continuous changes in the independent variable. This study revealed that even when some students accurately constructed graphs they had no conceptual understanding of what was actually occurring in the dynamic situations.

Implications:

These findings imply the need for instructors to more carefully monitor students' understanding and reasoning abilities in relation to dynamic situations. In light of this research instructors should place increased emphasis on moving students to the understanding of how the instantaneous rate of change with continuous change relates to the independent variable in dynamic function situations. This study also demonstrates the importance of students reflecting on their own understanding of changing rates of change and instantaneous rate of change in order to reveal their misunderstandings.

Galbraith, P. & Hanes, C. (2000). Conceptual mis(understandings) of beginning undergraduates. *International Journal of Mathematical Education in Science and Technology, 31*(5), 651-678.

Research Question:

- What type of coordination of algebraic and graphical representations do students have?
- What understanding do students have of parameterization?

Method:

This study had 423 beginning engineering and mathematics undergraduates from 1994-1996 answer a questionnaire that sought to reveal their mathematical foundation relating to algebraic and graphical representations. The three types of questions were categorized as mechanical, interpretive or constructive. The mechanical and interpretive questions were multiple-choice while the constructive ones required an extended response. The results were then analyzed to determine the relative difficulty of each item for the students.

Findings:

The results found that the students had difficulty analyzing the changing effect of parameters on algebraic forms and graphs. The students' ability to interpret graphically translations and scalings on given algebraic forms was poor. Meanwhile the students' connection between symmetry and algebraic forms was also weak. Students did well at graphically implementing algorithms of translation and scaling. Additionally, students conceptually understood the standard form of straight lines and its relationship to the graph. Finally students had a hard time adding and subtracting graphs. Overall the students performed the best on the mechanical performances and the worst on the constructive performances.

Implications:

These findings demonstrate the need for students to experience graphical manipulatives skills. Testing students' abilities to work with graphs and algebraic forms both forwards and backwards is essential to their foundation for undergraduate courses. It is not safe to assume that students understand what is actually occurring when they plug in equations into their graphing calculators. This study shows the importance of using technology to recognize and address the problems and misconceptions found in this study.

Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111-129.

Research Question:

- What do the use of examples in evaluating and proving conjectures reveal about the reasoning strategies used among expert mathematicians?

Method:

The two participants in this study were selected from a larger study about how highly talented mathematics students evaluate conditional statements. These participants were studying for a PhD in a prestigious UK university. Neither participant had prior experience with the concepts of abundant and deficient numbers. Each was interviewed for approximately one hour where they were presented first with some written information about abundant, perfect and deficient numbers. When the participants indicated they understood the information they were then given a written conjecture addressing these concepts and asked to determine, with proof, whether it was true or false. After the conjecture had been solved to the participant's satisfaction, the interviewer presented a new conjecture. When necessary, the interviewer would prompt the participants to clarify their methods.

Findings:

The research indicated that there was a striking difference between the strategies used by the two participants in proving the conjectures as being either true or false. One participant primarily used examples and informal rephrasing to determine the validity of the statements given. Meanwhile the second participant used primarily true known statements, standard proof frameworks and logical deductions to prove if the given conjecture was true or false. Neither was readily willing to adapt their initial approach using their preferred strategies even when the other participant's strategy was more appropriate for certain conjectures.

Implications:

These findings in relation to the different preferred strategies used have a couple implications for mathematics teaching. The model given in this report for evaluating student thinking is useful in any mathematics class where proving conjectures. The strategies used in this study are most applicable to undergraduate students because they are the ones next in line to become graduate students. If undergraduate students are to be taught the skills needed for success in advanced mathematics, it is important to know what those skills are and how they are employed by fluent mathematicians. Understanding the purpose of the different strategies will allow one to be more efficient and concise when seeking to prove or disprove a given mathematical statement. Finally, evaluating a student's reasoning using the framework put forth in this study will allow one to gain a better understanding of where that student is at in understanding the mathematical theory being discussed.

Chappel, K. K. (2006). Effects of concept-based instruction on calculus students' acquisition of conceptual understanding and procedural skill. *Research in Collegiate Mathematics Education, 6*(13), 27-60.

Research Questions:

- If an instructor chooses to emphasize the learning and practice of procedures and de-emphasize concept development, will students be able to perform well on examinations that require the extension of knowledge to new situations and that require students to explain verbally or graphically why a particular procedure makes sense?
- If an instructor chooses to emphasize concept development and de-emphasize the learning and practice of procedures, will students be able to perform well on examinations that require the recall and use of procedures and that require the extension of knowledge to new situations?

Method:

This study involved 144 calculus students and 4 instructors. Two instructors taught in the traditional way by emphasizing the learning and use of procedures while deemphasizing concept development. Meanwhile the other two instructors taught in a concept-based manner where concept development was emphasized and the learning and practice of procedures was deemphasized. A common midterm and final exam were used to measure the students' mastery of concepts and procedures. Additionally written questionnaires and classroom interviews were used to explore the students' perspectives regarding the concept-based instruction in relation to their acquisition of procedural skills and ability to extend their knowledge to unfamiliar problems.

Findings:

This research found that on both the midterm and final the students who were in the concept-based learning environments had a more thorough conceptual understanding and ability to solve unfamiliar problems. Through the questionnaires and classroom interviews the students in the concept-based learning environments communicated that the instruction was different from their previous mathematics classes and that it was at first frustrating and difficult. However there was an overwhelming consensus that they preferred this concept-based instruction while emphasizing that there must be a balance between understanding and practice of procedures. The students stated that when they understood why a procedure worked it was a lot easier to retain than just memorizing a formula and allowed them to make more connections when learning and applying new procedures.

Implications:

For instructors the results show the need to implement an effective balance between conceptual understanding and procedural skills. Through the student interviews and study's results the evidence implies the need for students to be taught why procedures work. In doing this it is important for instructors to recognize that students will initially resist being required to justify and explain why procedures work.

Anthony, G. (2000). Factors influencing first-year students' success in mathematics. *International Journal of Mathematical Education in Science and Technology, 31*(1), 3-14.

Research Question:

- What factors do undergraduate students and lecturers perceive as making the most important contributions to students' academic success in their first-year mathematics courses?

Method:

Qualitative data from 22 lecturers and 65 students was first collected through open-ended questionnaires. Next 92 students and 26 lecturers were asked to answer a questionnaire regarding how strongly they agreed or disagreed with 77 different statements regarding factors influencing student success. Finally, ten randomly pulled students participated in semi-structured interviews allowing more insight into the students' beliefs. The results were then analyzed and grouped according to the varying influencing factors.

Findings:

The findings demonstrated that both students and lecturers believed 'self motivation' was the most influential factor in determining students' success. The opinions diverged from this point however as lecturers tended to attribute student success as being most impacted by the incoming knowledge of students or the factors within the students' control. On the other hand students attributed factors relating to lectures and course design as being more influential in impacting student success. Additionally this study found that many of the students poor performance was due to ignorance about the study skills required in these courses or the inability to apply these skills appropriately.

Implications:

This research shows that understanding the discrepancy between student and instructors' beliefs relating to student success is crucial for instructors of undergraduate mathematics courses. Understanding how important students view lectures implies the opportunity for instructors to use the lecture times more strategically to stimulate student thinking. Additionally, for lecturers, the importance of communicating class expectations from the outset of a course will better equip students in knowing what their responsibilities are and how their instructor plans to help them succeed.

Trigueros, M. & Ursini, S. (2003). First-year undergraduates' difficulties in working with different uses of variable. *Research in Collegiate Mathematics Education, 5*(12)*,* 1-29.

Research Questions:

- What kind of understanding and ability do first-year undergraduates have in interpreting and using variables as unknowns, general numbers and variables in simple functional relationships?

Method:

The findings in this study were based off of 164 first-year undergraduate students who responded to a questionnaire consisting of 65 open-ended questions. Four students out of this group were additionally interviewed in finding evidence for students understanding and uses of variables. The results of the questionnaire were analyzed from a quantitative and qualitative perspective.

Findings:

The findings revealed that students' understanding of the different uses of variables is very weak. Many of the students were unable to discriminate between a variable being a specific unknown and a general number. The students' responses demonstrated that they often rely on and apply memorized rules without having a conceptual understanding of what the variables represented. Additionally the students sought to avoid manipulation of variables in algebraic expressions and relationship. Students were also unable to represent a general rule of an analytical expression.

Implications:

This research study implies the need for undergraduate courses to be designed to foster students' understanding of the concept of variable. These findings demonstrate that students often enter the universities with a very elementary understanding of variables. Identifying where students are at in relation to their understanding and ability to flexibly work with variables is crucial for instructors to be able appropriately provide them with support in conceptually understand variables and confidently working using simple functional relationships.

Szydlik, J. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education, 31*(3), 258-276.

Research Question:

- What are students' content belief and conceptual understanding of the limit of a function?
- What are the students' sources of conviction for their beliefs and understanding?

Method:

The subjects in this study were 27 university calculus students. They were chosen from 577 second-semester calculus students based off of their responses to a 20-item questionnaire used to elicit students' beliefs and sources for their understandings of the limit of a function. These questions addressed real numbers, infinity and functions as these concepts underlie the concept of limit. Based off of the students' answers the 27 primary participants were selected for a structured interview on limit conceptions. Nine PhD mathematics faculty members evaluated the scoring until a consensus was formed.

Findings:

The findings found that there was a correlation between students' content beliefs and sources of conviction. The students who had external sources of conviction, such as their instructor or textbook, had an increased number of misconceptions of limit as a bound and being unreachable. Meanwhile those who had internal sources of conviction through appealing to empirical evidence, intuition, logic or consistency provided more static and fewer incoherent definitions of limit. The results demonstrated that the students' perceptions of mathematics as being logically organized and interconnected were the ones with the internal sources of conviction. This research also found there was a direct correlation between students' understanding of function and limit. Additionally, many students had the misconception of a limit being a bound that cannot be reached or crossed. Meanwhile some of the students also did not have a correct understanding of real numbers in thinking that there are gaps in the real numbers. Students had trouble with the idea that every infinite decimal converges to a real number. This study demonstrated that even among second semester calculus students there are a wide range of beliefs about functions, real numbers and infinity.

Implications:

One of the implications is the importance of recognizing that if students are having a hard time with understanding the concept of limit there are most likely holes in their understanding of functions in general. The findings also demonstrate the need for instructors to teach a balance between allowing students to discover mathematical ideas and using mathematical arguments rooted in rigorous definitions. Additionally it is important for instructors to address the misconceptions discussed in the findings of this research in order to help students gain the correct conceptual understanding of limits.

Williams, S. R. (2001). Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education, 32*(4), 341-367.

Research Questions:

- What informal notions of limit do students have?
- What procedures and constructions do students bring to bear in understanding the concept of limit?

Method:

In this study two college calculus students' understanding of limit are studied and analyzed. These students met individually with an investigator over a seven week period for five sessions designed to challenge the common misconceptions students have about the limit concept. The findings are based primarily off of the first and fifth sessions where the students compared and contrasted different statements about a limit. These statements had common misconceptions about a limit in them and differed in the degree of their formality. The findings in the second through fourth sessions were used to clarify the student's comments in the first and fifth interviews.

Findings:

The research found that there was confusion for both subjects regarding whether or not a limit is actually ever reached. The students had a similar dynamic view of points being considered to get closer and closer to the limit value. Even though the experimental sessions were designed to reveal this misconception of limit involving motion the students continued to hold to this idea as being the fundamental way they understood limit. By the end of the experiment session both subjects seemed to refine their ability to distinguish between practical and theoretical aspects of limit. The findings reveal that the notion of reaching a limit rests on the fundamental distinction between actual and potential infinity.

Implications:

For instructors recognizing and understanding the way that the mathematical and cognitive approaches to limit are very different for students will allow them to accordingly address students' struggles to make sense of the limit idea. The implications are not addressed regarding the obstacle students have in understanding actual infinity. However it is important at this point for instructors to understand that this is a significant cognitive obstacle to learning the formal definition of limit.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500-507.

Research Questions:

- Do students understand the connections between algebraic and graphical representations of functions?

Method:

The participants in this study consisted of 178 students enrolled in 1st year algebra through calculus at a large suburban high school. These students were given one of six selected problems to solve. Each of these six problems required the understanding and use of equation-to-graph and graph-to-equation connections in determining a solution. The students were instructed to show all their work and explain their reasoning. Each problem gave an algebraic and graphical representation for a given function. The students were asked to provide an alternate solution method after their first selected method for solving the problem. The responses were then categorized as either algebraic or graphical solutions based off of the methods the students used to solve the problem.

Findings:

The results showed that more than 75% of the students chose an algebraic approach first even when a graphical approach would have been easier and more efficient. Overall less than a third of the students used a graphical solution as either their primary or alternative solution method. Additionally there was a consistent pattern in the students using one particular algebraic strategy to solve the problems.

Implications:

The findings in this study indicated that students' mastering the connection between graphs and equations is not as straightforward as is often assumed. It is important for students to understand which representation for a function is best. However this cannot be where a teacher stops. Developing a thorough and deep understanding of functions also means being able to interchange and work flexibly between the different representations. To help students master these connections it is important to give students opportunities to build connections between the algebraic and graphical representations through interacting with them.

Note: The following article is included because it provides most of the problems addressed in "Student understanding of the Cartesian connection: An exploratory study":

Knuth, E. J. (2000). Understanding connections between equations and graphs. *The Mathematics Teacher*, *93*(1), 48-53.

Thompson, P. W. (1994). Students, functions and the undergraduate curriculum. *Research in Collegiate Mathematics Education, 1*(4), 21-44.

Research Questions:

- What are students' understandings of functions?
- How is their understanding important for the undergraduate curriculum?

Method:

This paper combines many different research studies to discuss different aspects that affect students' understanding of functions. Six themes emerge from the research collected and the findings that emerge are then discussed accordingly.

Findings:

The findings in this study revealed that students often have a difficult time making the connections between definitions and images of what a function is. Students' understanding of the concept of function ranges from thinking of an expression as producing a result of calculating to the ability to reason about operations on sets of functions. Another theme that emerged was how students may view a function as elements in one set corresponding to elements in another set without recognizing that it represents a situation where one thing varies due to another (i.e.: temperature varies as time changes). Looking at the mathematics curriculum used in the schools there is no emphasis on the dynamic situations represented by functions. Another problem that emerged was that instructors erroneously assume students have a principled understanding of an underlying situation relating to a function while the students are actually inclined to just see a graph as a set of points.

Implications:

One implication from the research is the importance of giving explicit attention to the visual representations and mental pictures students have in relation to functions. The undergraduate curriculum is highly dependent upon a lot more conceptual development than what is being taught in the high schools. Developing the students' ability to think about the results and implications of a function and operations done to it is crucial. Recognizing that students enter college with minimal exposure to interpreting what the notation represents implies the need for instructors to address this problem. Students need to be oriented to understand and reason about dynamic situations and their constraints in order to then be able to accurately represent them as functions. It is crucial for instructors to recognize this disparity and then address the problem accordingly.

Linn, M. C., & Kessel, C. (1996). Success in mathematics: Increasing talent and gender diversity among college majors. *Research in Collegiate Mathematics Education, 2*(6), 101-144.

Research Question:

- Why do talented and diverse students who select mathematics as a college major switch to other fields?
- Why specifically are females more likely to switch to other fields of study either in their undergraduate or graduate study when across the United States they have a higher mathematics grade point average in both high schools and universities?

Method:

This research is based off of an accumulation of studies. These studies address a wide range of factors influencing the retainment of mathematics students at various institutions throughout the United States. The findings were found through interviews, questionnaires, and, in some cases, written reflections. Overall the data collected was from over 1400 participants in ten different studies.

Findings:

This study found that students' concerns with staying in the mathematics field entailed conceptual difficulties, preferring the teaching in non-mathematics courses and the curriculum being to fast paced or overloaded. Specifically for students who switch out of mathematics the reasons were the competitive classroom culture, other majors were more interesting and offered better education, there was inadequate advising and the teaching of the mathematics content was poor. Seemingly lack of career opportunities for mathematics majors also influenced students' decision to change majors. Several studies showed that males are expected to outperform females and females report lower expectations of their success. These stereotypes are seen as being reinforced by faculty. The most influential factor in students continuing to major in mathematics was the family and peer support they received

Implications:

One implication from this research was the need for students to conceptually understand the mathematics and to see the relevance of their work to other disciplines. Instructors need to understand what content is conceptually difficult for their students and intentionally address such difficulties. These studies revealed the need for instructors to create nurturing and encouraging environments of student interaction rather than promoting the idea of filtering out the seemingly weaker students. To improve mathematics instruction this study suggests that adding projects which require a deeper understanding will communicate a more robust and realistic view of mathematics by allowing students to see the importance of what they are learning. Another important implication is the need to pay attention to subtle messages being communicated when interacting with students that communicate bias and enforce stereotypes.

Leitze, A. R. (1996). To major or not major in mathematics? Affective factors in the choice-of-major decision. *Research in Collegiate Mathematics Education, 2*(6), 83-100.

Research Questions:

- What are some of the factors affecting undergraduate students' participation (or lack of participation) in mathematics?
- Specifically how are students' participation in mathematics affected by the perceived usefulness, asocial nature, difficulty and enjoyment of mathematics?

Method:

There were two parts to the study conducted. The first was a selection of 140 juniors and seniors at a university who had completed Calculus I, Calculus II or both for their mathematics/science majors. Two subgroups were formed of 70 students each and then paired with each other according to gender, calculus professor and calculus course grade. The subgroups were divided based off of whether or not the students had to take any 300+ level mathematics courses. These participants all answered 70 questions and the data was then analyzed and compared. For the second part of the study 18 respondents each had an hour long interview which was analyzed more in-depth upon its completion.

Findings:

The findings indicated that the usefulness of mathematics did not appear to be a consideration in undergraduates' selection of major. The college mathematics majors barely knew any professions that use a mathematics degree. Additionally, although it was widespread acknowledged that mathematics is hard, the research showed that this was not a factor in all but one of the participants' decision in what major they selected.

When it came to what did affect the students' selection of major the students' enjoyment of the subject was unanimously very important. Interestingly the data revealed that many students were unable to differentiate between enjoying the professor and enjoying the field of study. Also this study found that many students view mathematics as asocial. Only one subject had any experience of group work in the undergraduate program. All the participants described their mathematics learning in college as lacking professor-student interaction and void of group work.

Implications:

This study shows that piquing a student's interest in or enjoyment of a given discipline is a necessary but not sufficient condition to ensure that they will remain interested. These findings indicate that the lower division courses are vitally influential in determining undergraduate's choice of major. In light of the overwhelming consensus that math is asocial it appears that without changing this perception the level of student enjoyment will continue to remain limited. This is especially important to consider when enjoyment of the discipline was such a strong factor in the students' major selection. The research indicates that combining both lecture and collaborative methods in teaching will lead to more students enjoying the mathematics classrooms and more learning styles will be accommodated.

McDonald, M. A., Matthews, D. M., & Strobel, K. H. (2000). Understanding sequences: A tale of two objects. *Research in Collegiate Mathematics Education, 4*(8)*,* 77-102.

Research Question:

- What types of mental constructions do students make relating to their understanding of sequences?

Method:

The research conducted was based off of twenty-one students at a large Midwestern university who had each completed at least two semesters of calculus. Six of them completed the traditional courses and fifteen of them completed activity-based courses. In-depth individual interviews were conducted based off of fourteen questions. These recorded interviews were then analyzed by researchers until a consensus view of the student's understanding was reached.

Findings:

This study revealed that the students think a sequence entails a list of numbers. For some students this listing is merely a mathematical representation. One third of the students did not have a developed understanding of sequence as a function. Even with the students who did recognize this relationship for some there was still an absence of including the discrete nature of the domain. Most students were either unable to perform actions on functions or did not understand the implications of such actions. This in turn directly impacted their understanding of a sequence being a function. There was evidence of students being able to use a sequence as both a list and function. However, a rich understanding of their equivalence was present with only a third of the students. Finally, a problem arose in students seeing a bound as either a unique number or optima.

Implications:

For instructors this research implies the need to guide students to make the connection of equivalency between a sequence as a list and as a function. The study shows the need for students to be faced with new problem situations where there their inconsistent beliefs regarding the discrete nature of the domain of a sequence will be revealed. Additionally a better job emphasizing through examples that bounds are not unique and are often not equal to the limit will help students grow in their understanding of sequence. These findings indicate that the curriculum choices of instructors are crucial in guiding students to make these connections between the list and function as being one mathematical concept.

Herman, M. (2007). What students choose to do and have to say about use of multiple representations in college algebra. *International Journal of Mathematical Education in Science and Technology, 26*(1), 27-54.

Research Question:

- When presented with algebra problems related to polynomial, exponential and logarithmic functions and the freedom to use a graphing calculator, what solution strategies do students choose to use to compute the problems?
- What influences affect student choice of problem-solving strategy?
- How does a focus on use of multiple representations over the duration of a 10-week algebra course affect student ability to approach algebra problems with more than one problem-solving strategy and to successfully solve the problems?
- What beliefs do students hold about using multiple representations, particularly in terms of their understanding of functions?

Method:

This study was done in a freshman-level advanced algebra course at a large Midwestern university. In this course multiple approaches and strategies to problem solving were encouraged and explored relating to algebra and trigonometry problems. Thirty eight students from this ten week course were the participants. These students took pretest and posttest worksheets of six problems each. These answers were categorized into one of the following three categories based off of the students' method for solving the problems: symbolic manipulation, graphical strategy or tabular strategy. Seven of these students also participated in semi-structured interviews and teacher questionnaires.

Findings:

The findings found in this study demonstrated that students view algebraic manipulation as being more 'mathematical'. The graphical method was used but some students shared that they saw this as cheating. Meanwhile the tabular method was never used by itself. In the interviews the students communicated that their instructors appeared to stress the algebraic manipulation the most and that the other two methods were used as a back up to check their work. They acknowledged their instructors also seemed to like the graphical strategy. Most of the students saw the tabular strategy as being unnecessary. This study found that the students who used all three strategies were most likely to get the correct answer. Some of the students also communicated that learning the different strategies through the course built their confidence and helped them see more connections between the three representations.

Implications:

For instructors this study implies the need to help students see the connections between the algebraic manipulation, the graphical method and the tabular method. Helping students understand that all three are valid and accurate ways of solving algebra and trigonometry problems will increase student confidence. This study demonstrated the necessity of ensuring that students are comfortable with using graphing calculators and tables in solving problems. Additionally this research implies the importance of helping students use all three methods in order to ensure that their answers are correct.

Classroom Lessons

- **A Conceptual Approach to Solving Equations**

  Swain, S. G. (1990). A conceptual approach to solving equations. *Mathematics Teacher, 83*, 454-456.

- **A Deeper Look at Related Rates in Calculus**

  Santulli, T. V. (2006). A deeper look at related rates in calculus. *Mathematics Teacher, 100*(2), 126-130.

- **A Tool to Use the First Day of Calculus**

  Van Dyke, F. & White, A. (2004). A tool to use the first day of calculus. *PRIMUS, 14*(3), 213-226.

- **Another Way to Graph a Sequence**

  Olson, D. (1996). Another way to graph a sequence. *The College Mathematics Journal, 27*(3), 208-209.

- **Average and Instantaneous Velocity**

  Ventress, A. (2006). Average and instantaneous velocity. *Mathematics Teacher, 100*(1), 56-58.

- **Average Rate of Change**

  Pleacher, D. H. (1992). Average rate of change. *Mathematics Teacher, 85*(6), 445-446.

- **Bicycles, Birds, Bats and Balloons: New Applications for Algebra Classes**

  Yoshiwara, B. & Yoshiwara K. (1992). *Bicycles, birds, bats and balloons: New applications for algebra class*. Paper presented at the Annual Meeting of the American Mathematical Association of Two-Year Colleges, Chicago, IL.

- **Building an Understanding of Functions: A Series of Activities for Pre-Calculus**

  Carducci, O. M. (2008). Building an understanding of functions: A series of activities for pre-calculus. *PRIMUS, 18*(4), 394-397.

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Classroom Lessons Matched with the Research Articles

A Longitudinal Study of the C^{4}L Calculus Reform Program:
Comparisons of C^{4}L and Traditional Students

Schwingendorf, K.., McCabe, G., & Kuhn, J.

- **A Conceptual Approach to Solving Equations**
- **A Deeper Look at Related Rates in Calculus**
- **Average Rate of Change**
- **Building an Understanding of Functions: A Series of Activities for Pre-Calculus**
- **Culture Points: Engaging Students Outside the Classroom**
- **Finding Volumes with the Definite Integral: A Group Project**
- **Hands-On Calculus**
- **Is the Derivative of a Product the Product of the Derivatives?**
- **Rate of Change of Exponential Functions: A Precalculus Perspective**

A Theoretical Framework for Analyzing Student Understanding of the Concept of Derivative

Zandieh, M. J.

- **Average and Instantaneous Velocity**
- **Bicycles, Birds, Bats and Balloons: New Applications for Algebra Classes**
- **Calculus: An Active Approach with Projects**
- **Derivatives Project**
- **Instantaneous Rate of Change: A Numerical Approach**
- **Rate of Change of Exponential Functions: A Precalculus Perspective**

Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E.

- **A Conceptual Approach to Solving Equations**
- **A Deeper Look at Related Rates in Calculus**
- **Building Connections among Classes of Polynomial Functions**
- **Precalculus Explorations of Function Composition with a Graphing Calculator**

Conceptual Mis(understandings) of Beginning Undergraduates

Galbraith, P. & Hanes, C.

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- **Building an Understanding of Functions: A Series of Activities for Pre-Calculus**
- **Building Connections among Classes of Polynomial Functions**
- **Connecting Procedural and Conceptual Knowledge of Functions**
- **Exploring Functions: A Calculator Game**
- **Graphing Families of Curves Using Transformations of Reference Graphs**
- **Helping Students Connect Functions and Their Representations**
- **Multiple Representations for Pattern Exploration with the Graphing Calculator and Manipulatives**
- **Precalculus Explorations of Function Composition with a Graphing Calculator**

Doctoral Students' Use of Examples in Evaluating and Proving Conjectures

Alcock, L., & Inglis, M.

Effects of Concept-Based Instruction on Calculus Students'

Acquisition of Conceptual Understanding and Procedural Skill

Chappel, K. K

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- **Derivatives Project**
- **Edible Calculus**
- **Finding Volumes with the Definite Integral: A Group Project**
- **Hands-On Calculus**

Factors Influencing First-Year Students' Success in Mathematics

Anthony, G.

First-year Undergraduates' Difficulties in Working with Different Uses of Variable

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- **A Conceptual Approach to Solving Equations**
- **Building an Understanding of Functions: A Series of Activities for Pre-Calculus**
- **Connecting Procedural and Conceptual Knowledge of Functions**
- **Multiple Representations for Pattern Exploration with the Graphing Calculator and Manipulatives**

Mathematical Beliefs and Conceptual Understanding of the Limit of a Function

Szydlik, J.

- **Card Folding: An Investigation with Limits**
- **Getting Limits Off The Ground Via Sequences**
- **Introduction to Limits, or Why Can't We Just Trust the Table?**

Predications of the Limit Concept: An Application of Repertory Grids

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Student Understanding of the Cartesian Connection: An Exploratory Study

Knuth, E. J.

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- **A Tool to Use the First Day of Calculus**
- **Bicycles, Birds, Bats and Balloons: New Applications for Algebra Classes**
- **Building an Understanding of Functions: A Series of Activities for Pre-Calculus**
- **Building Connections among Classes of Polynomial Functions**
- **Connecting Procedural and Conceptual Knowledge of Functions**
- **Exploring Functions: A Calculator Game**
- **Graphing Families of Curves Using Transformations of Reference Graphs**
- **Helping Students Connect Functions and Their Representations**

Students, Functions and the Undergraduate Curriculum

Thompson, P. W.

- **A Deeper Look at Related Rates in Calculus**
- **A Tool to Use the First Day of Calculus**
- **Building an Understanding of Functions: A Series of Activities for Pre-Calculus**
- **Building Connections among Classes of Polynomial Functions**
- **Connecting Procedural and Conceptual Knowledge of Functions**
- **Exploring Functions: A Calculator Game**
- **Graphing Families of Curves Using Transformations of Reference Graphs**
- **Helping Students Connect Functions and Their Representations**
- **Multiple Representations for Pattern Exploration with the Graphing Calculator and Manipulatives**
- **Using Graphs to Introduce Functions**

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Linn, M. C., & Kessel, C.

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To Major or Not Major in Mathematics? Affective Factors in the Choice-of Major Decision

Leitze, A. R.

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- **Bicycles, Birds, Bats and Balloons: New Applications for Algebra Classes**
- **Building an Understanding of Functions: A Series of Activities for Pre-Calculus**
- **Calculus: An Active Approach with Projects**
- **Connecting Procedural and Conceptual Knowledge of Functions**
- **Culture Points: Engaging Students Outside the Classroom**
- **Exploring Functions: A Calculator Game**
- **Graphing Families of Curves Using Transformations of Reference Graphs**
- **Hands-On Calculus**

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McDonald, M. A., Matthews, D. M., & Strobel, K. H.

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What Students Choose to Do and Have to Say About

Use of Multiple Representations in College Algebra

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- **Building Connections among Classes of Polynomial Functions**
- **Connecting Procedural and Conceptual Knowledge of Functions**
- **Exploring Functions: A Calculator Game**
- **Helping Students Connect Functions and Their Representations**
- **Multiple Representations--Using Different Perspectives to Form a Clearer Picture**
- **Using Graphs to Introduce Functions**