The Convergence Rates of Quadrature Rules for Non-Smooth Data and Non-Smooth Functions

Midterm Report

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Error estimates for computations of integrals can be derived by viewing quadrature rules as Riemann-Stieltjes integral approximations, as explained in the proposal. The idea behind this method of error estimation is to estimate the integrator rather than the integrand, which is often done in Calculus courses. The integrators used for estimations are step functions whose derivatives are linear combinations of delta functions. As mentioned in the proposal, this project will emphasize the use of the midpoint rules.

Consider the composite midpoint rule for a continuous function f on the interval from a to b. The integrator *α(x)* can be defined as follows:

$$\alpha(x) = \begin{cases} 0 & a \le x < a + h/2 \\ h & a + h/2 \le x < a + 3h/2 \\ 2h & a + 3h/2 \le x < a + 5h/2 \\ . & . \\ . & . \\ . & . \\ (n-1)h & a + (2n-1)h/2 \le x < b \end{cases}$$

Note that *h = (b-a)/n* where n is the number of subdivisions. Suppose *B(x) = x – a then dB(x) = dx*. The integration by parts formula for Riemann-Stieltjes integration gives

$\int_a^b f(x)dx = h\sum_{i=1}^{N} f(x_i) + \int_a^b [\alpha(x) - B(x)]f'(x)dx$, an identity that is easy to check without Riemann-Stieltjes theory. Note that the sum is the composite midpoint rule for $x_i = (2i-1)h/2 + a$ and the integral on the right hand side of the equation is the error. Since $\alpha(x)$ is a step function, the error can be rewritten as

$$\int_a^b [\alpha(x) - B(x)]f'(x)dx = \int_a^{a+h/2} [\alpha(x) - B(x)]f'(x)dx + \int_{a+h/2}^{a+3h/2} [\alpha(x) - B(x)]f'(x)dx + \ldots \int_{a+(2n-3)h/2}^{a+(2n-1)h/2} [\alpha(x) - B(x)]f'(x)dx$$

A second problem explored in this project is to show that, for the midpoint rule,

$\left| \int_a^b f(x)dx - h\sum_{i=1}^{N} f(x_i) \right| \le Mh(b-a)/4$ if f'(x) is bounded by a constant M. Note that all of the variables are defined as before. We know that $\int_a^b f(x)dx = \sum_{i=1}^{N} \int_{a+(i-1)h}^{a+ih} f(x)dx$, and this sum is equal to the composite midpoint rule plus the error as shown in the first problem. Therefore,

$\int_a^b f(x)dx = h\sum_{i=1}^{N} f(x_i) + \sum_{i=1}^{N} \int_{xi}^{xi+h/2} (x_i + h/2 - x)f'(x)dx + \sum_{i=1}^{N} \int_{xi-h/2}^{xi} (x_i - h/2 - x)f'(x)dx$. Clearly the error is the difference between the exact value of the definite integral and the results obtained from the midpoint rule. The difference can be written as follows:

$$\left| \int_a^b f(x)dx - h\sum_{i=1}^{N} f(x_i) \right| \le \sum_{i=1}^{N} \left| \int_{xi}^{xi+h/2} (x_i + h/2 - x)f'(x)dx \right| + \sum_{i=1}^{N} \left| \int_{xi-h/2}^{xi} (x_i - h/2 - x)f'(x)dx \right|$$

If $|f'(x)| \le M$, then clearly

$$\left| \int_a^b f(x)dx - h\sum_{i=1}^{N} f(x_i) \right| \le \sum_{i=1}^{N} \left| \int_{xi}^{xi+h/2} (x_i + h/2 - x)Mdx \right| + \sum_{i=1}^{N} \left| \int_{xi-h/2}^{xi} (x_i - h/2 - x)Mdx \right|$$

The right hand side evaluates to

$\sum_{i=1}^{N}[\,|M*(x_i-h/2-x)^2/2\,|_{xi-h/2}^{xi} - |M*(x_i+h/2-x)^2/2\,|_{xi}^{xi+h/2}]$, which simplifies to

$2NMh^2/8$. Since h = (b-a)/N, the error is bounded by Mh (b-a)/4.

A third problem is to study the convergence of the error of midpoint rule for the non-smooth function $f(x) = x^p$ for $0< p<1$. According to the midpoint rule,

$$\int_a^b x^p dx = h\sum_{i=1}^{n}((2i-1)h/2)^p + \sum_{i=1}^{n}\int_{(i-1)h}^{(2i-1)h/2}[(i-1)h-x]px^{p-1}dx + \sum_{i=1}^{n}\int_{(2i-1)h/2}^{ih}[ih-x]px^{p-1}dx$$

The error can be written as

$$\int_0^{h/2}(-x)px^{p-1}dx + \int_{h/2}^{h}(h-x)px^{p-1}dx + \sum_{i=2}^{n}\int_{(i-1)h}^{(2i-1)h/2}[(i-1)h-x]px^{p-1}dx + \sum_{i=2}^{n}\int_{(2i-1)h/2}^{ih}[ih-x]px^{p-1}dx\,.$$

Integrate and we get

$$-p[x^{p+1}/(p+1)]_0^{h/2} + p[hx^p/p - x^{p+1}/(p+1)]_{h/2}^{h} +$$

$$\sum_{i=2}^{n}[(i-1)hx^p - px^{p+1}/(p+1)]_{(i-1)h}^{(2i-1)h/2} + \sum_{i=2}^{n}[ihx^p - px^{p+1}/(p+1)]_{(2i-1)h/2}^{ih}$$

Through algebraic simplifications we get $h^{p+1}(1/(p+1) - 1/2^p) + h^{p+1}\sum_{i=2}^{n}G(i)$ where

$G(i) = i^{p+1}/(p+1) * [1 - (1-1/i)^{p+1}] - i^p(1-1/(2i))^p$. By the binomial expansion,

$(1-1/i)^{p+1} = 1 - (p+1) + p(p+1)/(2i^2) + \ldots$

and

$(1-1/(2i))^p = 1 - p/(2i) + (1/2)\,p(p-1)(1/(2i))^2$.

Plugging these expansions into G(i) and simplifying, we get

$G(i) = p(p-1)i^{p-2}/24 - (p-1)(p-2)i^{p-3}/48 + \ldots$

Clearly, $G(i) \leq i^p \sum_{i=2}^{\infty} 1/i^n = i^p \sum_{i=0}^{\infty} (1/i)^n$. We know that the infinite series here is the geometric series and thus $G(i) \leq i^{p-2} * i/(i-1) = i^{p-1}/(i-1)$. We know that the error converges if $\sum_{i=2}^{\infty} G(i)$ converges. By the comparison test of the limits, $\lim_{i->\infty} i^{p-2}/(i^{p-1}/(i-1)) = 1$ implies that $\sum_{i=2}^{\infty} i^{p-1}/(i-1)$ converges since we know $\sum_{i=2}^{\infty} i^{p-2}$ converges. Therefore $\sum_{i=2}^{n} G(i)$ converges and thus the error converges.

With these three problems completed, we intend to work on two additional problems for the rest of the semester. The first problem we will work on is to investigate the convergence of the error of the midpoint rule for the function $f(x) = x^p$ for $-1<p<0$. We want to know whether the error will converge like for the case when $0<p<1$. The second problem we will work on will be to investigate the speeding up of the convergence of the midpoint rule by the use of higher order integration by parts. We intend to mimic the Romberg procedure for the trapezoidal rule in this process, and we expect to take the rest of the semester to complete these two problems.