MATH 215 – MATLAB Project

Project developed by: Jessica Bernier and William Yslas Vélez

Investigating the behavior of z^n as $n \to \infty, n \in N$ and z is a complex number (N denotes the positive integers.)

We will investigate the convergence of a sequence of the form z^n. A heuristic definition of convergence is the following: The sequence z^n converges to L if z^n gets closer to L as $n \to \infty$.

Investigation 1: We first consider the case when $z = a$, where a is a real number. So, we ask: If a is a real number, what happens to a^n as $n \to \infty, n \in N$?

Let's consider different cases of $|a|$. For each of the following cases, what can be said about a^n as $n \to \infty, n \in N$? Does the sequence converge? If not, can you describe the behavior of this sequence (Hint: In cases iii) and iv) you might distinguish between n odd and n even)?

i) $|a| < 1$

ii) $a > 1$

iii) $a < -1$

iv) $|a| = 1$

Investigation 2: Now let's consider the case for a complex number z of the form $z = a + bi$ (where a and b are real numbers.) The behavior of z^n as $n \to \infty$ will similarly depend on $|z|$. Using your answers to Investigation 1 and your intuition about the complex plane, what do you expect will happen to z^n as $n \to \infty$ for these cases of $|z|$?

i) $|z| < 1$

ii) $|z| > 1$

Investigation 3: We will examine the case $|z| = 1$ more carefully to develop a conjecture as to the behavior of z^n as $n \to \infty$. (Note: you have already investigated the behavior of z^n as $n \to \infty$ for the special cases of $z = \pm 1$. This investigation deals with any complex number that satisfies $|z| = 1$.)

Two MATLAB programs are provided, "converge.m" and "converge1.m". Run "converge.m" to help you answer the next few questions. Hints on running the program correctly are on the last page.

i) With $z = a + bi$ and $|z| = a^2 + b^2 = 1$ we need to chose values of a between -1 and 1. The program will then compute the appropriate value of *b*. Run the program for several values of *a*. What is it that you observe? What would you conjecture about the set $\{ z^n : n \in N \}$? That is, when you graph the values z^n, for positive integer values, what will the graph look like?

Note: you have already investigated the behavior of z^n as $n \to \infty$ for the special cases of $z = \pm 1$.

Conjecture:

ii) In the above conjecture, did you take into account what happens when *z* is either 1 or -1? In those two cases, you obtain periodic behavior. Are there any other values of *z* for which there is periodic behavior? In order to investigate periodic behavior, it will be useful to use a different representation of a complex number. An alternative notation for complex numbers is the polar form:

$$z = r\exp(2\pi i\theta)\text{, where } r = |z| \text{ and } \theta = \tan^{-1}(b/a)/2\pi \in [0,1]$$

Usually, when a complex number is expressed in polar form, θ can be a chosen to be a value between *0* and 2π. By dividing by 2π in the above expression, the value of θ lies between *0* and *1*. Of course,

$$z = r\exp(2\pi i\theta) = r\cos(2\pi\theta) + ir\sin(2\pi\theta).$$

When you are asked to input "*a*", you will now input $\cos(2\pi\theta)$.

Given that

$$\exp(2\pi i\theta)^n = \exp(2\pi in\theta)\text{, and } \exp(2\pi im) = 1 \text{ for any integer } m,$$

what values of θ will create periodic behavior of z^n?

Use MATLAB programs (consider using converge1.m) to help you in formulating a conjecture. MATLAB has a “zoom” feature. Use this feature to zoom in on the graphs to get a close up view of how the numbers appear on the graph.

Conjecture: The values of θ that yield periodic behavior are:

Can you think of a proof for this conjecture?

Investigation 4: We began this project by investigating the behavior of the sequence: z^n as $n \to \infty, n \in N$. If $|z| = 1$, for what values of z do you think this sequence will converge?

Conjecture:

Hints for running the programs:

Here are some tips to help you run the code correctly:

Copy the program into your MATLAB editor. Once you have copied the program into the editor, click on the icon with the green arrow ("debug") to run the program. Once the program starts you will be prompted for a value of *a*. To run a new value of *a*, either press the debug button again, or type in the name of the program into the command window and hit enter.

The script will prompt you to enter the real component, *a*, then it will calculate a value for the imaginary component, *b*, such that the condition $|z| = \sqrt{a^2 + b^2} = 1$ is satisfied.

To formulate a conjecture in Investigation 3, part ii), you are asked to input $\cos(2\pi\theta)$ when the program asks you for *a*. For this, you should type out "cos(2*pi*theta)," where you will actually input the value of θ that you want to use (e.g. for $\theta = 0.5$ you will type "cos(2*pi*0.5)".)

The script will plot 500 points on the complex plane, beginning with z^1, showing you the position of z^n for each successive *n*. If your computer is fast enough, this should look like a movie, so you can tell which order the points appear on the graph by watching carefully.

The first program, "converge.m" will plot all of the points in one color, while "converge1.m" will add color to help you better distinguish the behavior of z^n as $n \to \infty$. The first 100 points (z^1- z^{100}) are plotted in blue; the next 100 points in red, and so forth. This will be useful in determining periodic behavior (how will you know if the behavior is periodic?)

We recommend waiting until the plot is finished being generated before you close the figure window – this occurs when a title and axis labels appear on the figure, and the message "All done!" appears in the command window.