On a Property of Cosets in a Finite Group

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Let us fix the following notation. Let $G$ be a finite group, $H$ a normal subgroup of $G$, $g \in G$. Choose $b \in gH$ such that $o(b) \leq o(gh)$, for all $h \in H$, where $o(\cdot)$ denotes the order of that element in $G$.

In an investigation dealing with radical extensions of fields \cite{1}, we showed that if $G$ is abelian, then $o(b)|o(gh)$, for all $h \in H$.

At a recent A.M.S. meeting, we asked I. Kaplansky if he had ever seen this result. Kaplansky had not and in turn asked if this result held for a wider class of groups. Though this result does generalize somewhat, as the following theorems show, the result does not generalize to all groups. The example at the end of this paper is due to H. W. Lenstra, Jr.

**Theorem 1.** If $H$ is nilpotent, then $o(b)|o(gh)$, for all $h \in H$.

**Proof.** Let $H = P_1 \cdots P_s$, where each $P_i$ is the unique $p_i$-Sylow subgroup of $H$. Set $n = o(gH)$ in $G/H$. Then $g^n \in H$, so $g^n = g_1 \cdots g_s$, $g_i \in P_i$.

Since the $g_i$ all commute with each other and their orders are relatively prime, we have that $o(g) = n \cdot \prod_{i=1}^s o(g_i)$.

If $h \in H$, then $h = h_1 \cdots h_s$, $h_i \in P_i$, and $(gh)^n = g^n h^{e_{i-1}} h^{e_i-2} \cdots h^{e_1} = \prod_{i=1}^s h_i$, where $h_i = g_i \prod_{k=0}^{e_i-1} h_i^{e_i} \in P_i$.

Thus $o(gh) = n \prod_{i=1}^s o(h_i)$.

Choose $\bar{h}_i \in P_i$ such that $g_i \prod_{k=0}^{e_i-1} \bar{h}_i^{e_i}$ has minimal order among all elements of the form $h_i$. Since these elements have orders a power of $p_i$, the minimal order divides the orders of the other elements. Thus if we set $b = g(\bar{h}_1 \cdots \bar{h}_s)$, $b$ has minimal order and its order divides the orders of the other elements in the coset $gH$.

**Theorem 2.** If $o(gH) = p^r$ in $G/H$, $p$ a prime, then $o(b)|o(gh)$, for all $h \in H$.

**Proof.** Let $g^p = h \in H$ and $o(h) = p^m$, $(p, m) = 1$, then $o(g) = p^{r-1}m$.

Since $(m, p) = 1$, $\langle gH \rangle = \langle g^mH \rangle$ and $o(g^m) = p^{r-1}$. Choose $k$ so that 412

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\[ g^{mk} H = gH, \] where of course \((k, p) = 1\). Then \(g^{mk} \in gH\) and \(o(g^{mk}) = p^{r+1}\).

Thus we have shown that the order of every element in \(gH\) is divisible by \(p^r\) and if \(o(gh) = p'm, \ (p, m) = 1\), then there exists an \(h' \in H\) such that \(o(gh') = p^r\). Now let \(b \in gH\) be such that \(o(b) = p^r\) and \(r\) is minimal. Clearly \(p^r | o(gh)\) for all \(h \in H\).

**Example.** Let \(H\) be the semidirect product of a cyclic group \(C_9\) of order 9 with the quaternion group \(Q_8\) of order 8, with \(C_9\) acting on \(Q_8\) via an automorphism of order 3. It is easy to see that the center of \(H\) is cyclic of order 6. Thus let \(Z(H) = \langle a \rangle\), where \(a^6 = 1\). Now let \(G = \langle H, g \rangle\), where \(g^6 = a\) and \(gh = hg\), for all \(h \in H\). It can then be shown (the author used CAYLEY) that the coset \(gH\) contains elements of orders 12 and 18 but no element of order 6.

**References**