Square Tiles and the Fundamental Theorem of Arithmetic
by

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Introduction

Square tiles are a useful tool for introducing topics in mathematics classrooms. They provide a method for students to model a problem geometrically and permit a student to make visual connections with topics in mathematics.

The Fundamental Theorem of Arithmetic states "Every positive integer greater than one can be expressed as a product of primes". Note that even prime numbers satisfy this theorem since a prime can be expressed as a product of one prime, namely itself. This article will describe a lesson that makes use of square tiles to introduce this theorem to middle school students. With the square tiles, this lesson provides examples where geometric reasoning transitions into algebraic reasoning. The lesson allows students the opportunity to discover and investigate patterns in the data that they have constructed. These patterns lead to important research problems in number theory. One of the goals of this paper is to show students that there are problems in mathematics to which we do not know the answers (as of yet).

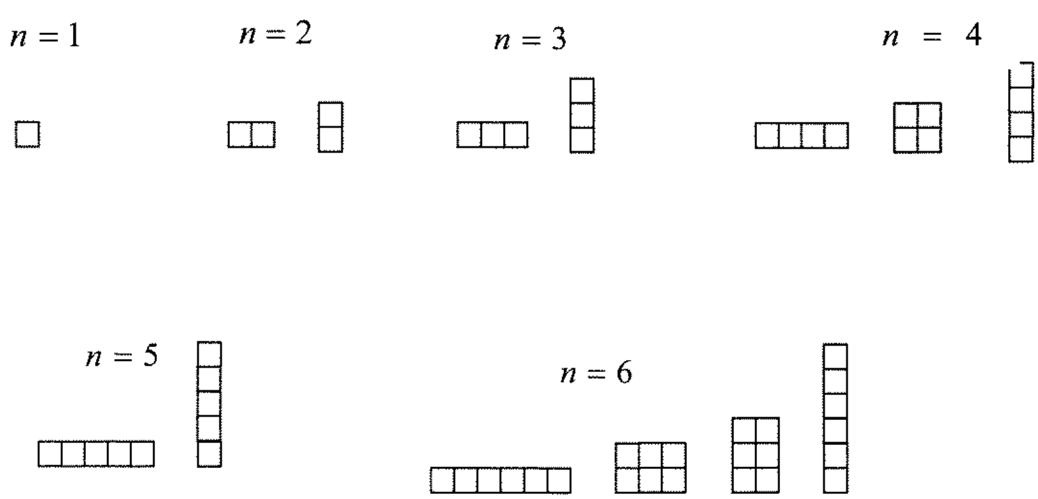
Lesson Description

In this lesson, students had access to a set of square tiles that contained at least 25 tiles. For a given positive integer n (up to 25), the squares were used to determine the number of rectangles that can be built with n squares. The lesson began by having the students consider how many rectangles could be formed from one tile. (A discussion on whether a square is a rectangle or a rectangle a square may be appropriate at this time.) Next, the students were asked how many rectangles could be built with two squares. They needed to differentiate between the 1×2 rectangle and the 2×1 rectangle. This process continued for $n = 1, 2, 3, \ldots, 25$ squares. The students kept track of this information by drawing pictures on graph paper of each rectangle formed with n squares and creating a chart, where they plotted the ordered pair

$$(n, \text{Number of rectangles using exactly } n \text{ squares}).$$

For the first few values of n, the students obtained the following pictures (Seen in figure 1):

Figure 1



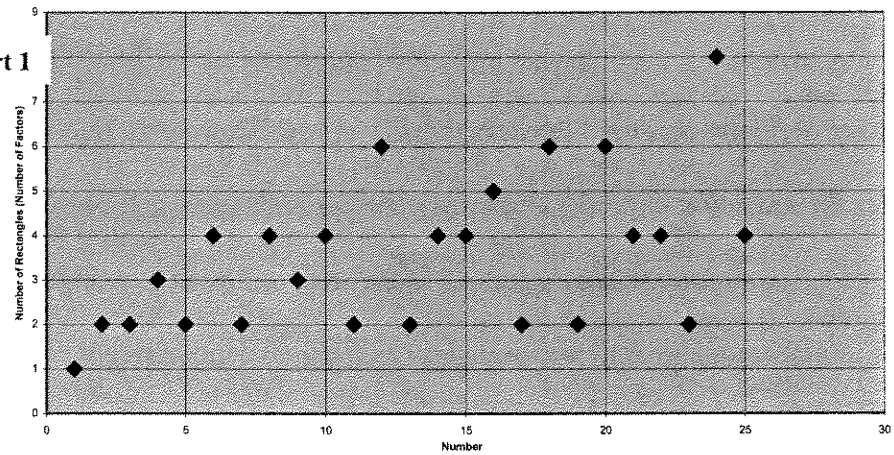
From this information the following ordered pairs were plotted on the chart: (1,1), (2,2), (3,2), (4,3), (5,2) and (6,4).

Students were informed of the useful mathematical technique of associating a complicated object with a number. The rectangles that the students could build for numbers greater than 25 would be large and cumbersome. This was demonstrated by getting students into groups of four and asking them to build all rectangles for n=82 and n=100. After allowing the students some time on this task, they were reminded that it might be easier for them to use numbers instead of the rectangles. It was suggested to the students that they should assign each rectangle a number, namely, the height of the rectangle. Students were asked if they recognized the height of each rectangle represented for n = 1, 2, 3, …, 25 (the number of tiles). For example, when n=6, the numbers assigned to the rectangles would be 1, 2, 3, 6. The students were led to the idea that the numbers were the factors of n. They were also informed that in mathematical terminology, they had created a one-to-one relationship between the factors of n and the rectangles that are made up of n squares. This is an early example of the important connection between algebraic thinking and geometric thinking. The abstract concept of the number of factors of a number is represented quite strikingly in this geometric representation.

The students were asked to find ways they could multiply two numbers to get 82. There are only two ways to do this: 1×82 and 2×41. The students then determined the number of rectangles that could be formed for n=82. They found that there are different configurations of rectangles with heights of 1, 2, 41 and 82. Similarly, for n=100, the students were asked for ways to multiply two integers to get 100. They are 1×100, 2×50, 4×25, 5×20 and 10×10. This resulted in rectangles with labeled heights 1, 2, 4, 5, 10, 20, 25, 50 and 100.

The class returned to the chart built by the students for n=1, 2, 3, …., 25.

Chart 1



A few of the chart's features were discussed with the students. First, there is the fact that $n=1$ has only one rectangle that corresponds to it. For any other value of n, students can form at least two rectangles, the $1 \times n$ and the $n \times 1$ rectangles. The number 1 is special. Throughout the students' mathematical studies, the special properties of 1 will arise again and again.

Another notable feature of the chart was that for some values of n there were only two rectangles formed by n squares. Students were asked to identify what was special about these types of numbers. After some discussion, they recognized these numbers as being prime numbers. A number n is a prime, when it is divisible only by 1 and itself. If this was interpreted geometrically, n is a prime number if and only if exactly two rectangles can be built with n squares. Students now have a geometric interpretation of this algebraic concept.

The importance of prime numbers was emphasized to the students. The height of each rectangle that they built was either a prime number or a product of prime numbers. (This is a consequence of the Fundamental Theorem of Arithmetic.) If the chart were to be continued, students were asked to consider whether there would always be values of n that have 2 as the second coordinate? After some discussion, the class was informed that the answer is yes, since there are an infinite number of primes.

Students were asked to consider constructing the chart past 25. The process to build the simple graph produced many interesting relationships and posed some arithmetic difficulties and questions. How could the class efficiently construct the data for this graph? How could they be sure they found all of the rectangles for a specific number? Did they want to use the square tiles to do so? Students were asked to model a few examples, $n = 6$ and $n = 15$. (Seen in figure 2 and figure 3 below.)

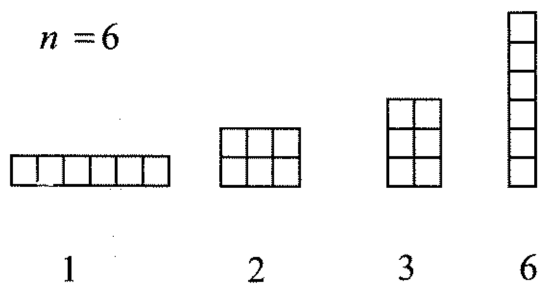


Figure 2

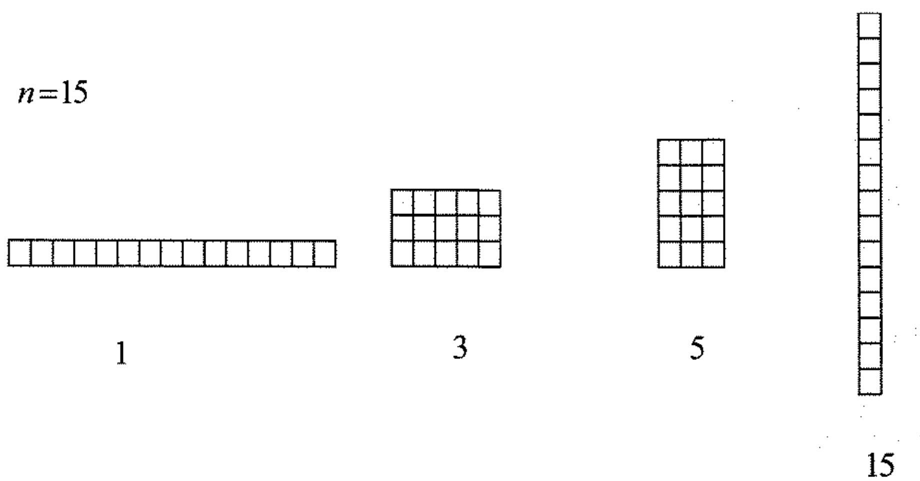


Figure 3

The class found that in both cases, four rectangles were obtained, or algebraically, each of the two numbers had exactly four factors. The students were asked to find similarities between the numbers 6 and 15. After some discussion it was found that each one of the two numbers is the product of two different primes, that is, 6=2*3 and 15=3*5. The four factors of 6 are: 1, 2, 3, and 6 and the four factors of 15 are 1, 3, 5 and 15. Students were able to identify that any number that is the product of two primes will have four factors. This is exactly what was done earlier when the rectangles for the number 82 were built.

At this point in the lesson, students were told (if they hadn't discovered it for themselves) that building the rectangles using the square tiles is a great way to illustrate the numeric properties of "small" numbers. But it would be an impractical and tedious method to use when considering a large number. Students were told that once they began assigning numbers to the rectangles they had moved into the algebraic domain, when initially it began as a simple geometric problem. Students were also told that there is another very important reason the students should move to the algebraic domain. The rectangles hide the essential feature of the calculation. For example, the number 6(= 2*3) and the number 1333 (=31*43) each have four factors. This is a result of the two numbers having the same algebraic structure. They share the same form of prime factorization or in other words they are the product of two different primes. The class was asked to imagine what it would be like to try to figure out how many rectangles could be formed with 1333 squares. The job would be horrible! By changing a geometric problem into an algebraic problem, we can transform a time consuming problem into an easy one.

The class was asked if there were any values of n for which the second coordinate is 3 or equivalently a number n that has three factors? The students were able to identify these numbers in the early part of their chart. They first appear for $n = 4 = 2^2$ and $n = 9 = 3^2$, not $n = 16 = 4^2$, and again for $n = 25 = 5^2$. Students were asked why they thought this pattern existed. The class explored the question by looking at the rectangles for each of the perfect squares and their prime factorizations. After some discussion the class agreed that when the prime factorization of n has the form of the square of a prime its only factors are1, p and p^2 and the second coordinate would be a 3.

The students were then asked to find values for n where the second coordinate is 5 or a number that has 5 factors. They were able to identify on their graph that $n = 16 = 4^2$ has 5 factors. The class explored the idea of the prime factorization of 16 and the rectangles built from 16 squares to determine what other numbers would have a second coordinate of a 5. For example, the prime factorization of $16 = 2^4$ and the rectangles built are shown in figure 4.

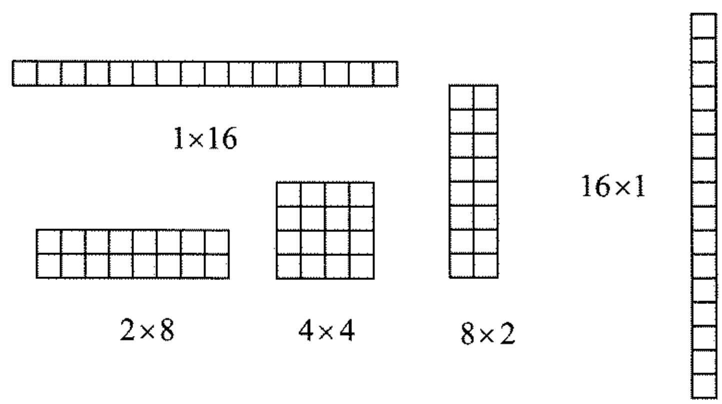


Figure 4

The height of each rectangle corresponds to a factor of 16. The class was asked to determine the number of factors of $81 = 3^4$ by considering the previous example. The factors turned out to be 1, 3, 9, 27, 81.

The class was able to conjecture that when the prime factorization of n has the form p^4, where p is a prime number, the second coordinate would be 5. Or the number would have 5 factors ($1, p, p^2, p^3, p^4$).

The class was then asked to determine the number of factors of a number of the form p^5 by considering the example $32 = 2^5$. Using the number 32, the students saw that the factors of the number were $1, 2, 2^2, 2^3, 2^4, 2^5$ or six factors. Using this idea the class was asked to determine the factors of p^5. By comparing the last example students were able to find that the number would have six factors.

Using these ideas one could build a chart for the number of factors of n up to 50. This graph was presented to the students. The prime numbers were highlighted on the chart. The students were asked if there were any patterns that they could identify. A few students noticed the pairs, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43. The students were asked that if the graph were to go on forever would we always be able to find primes that were only one number apart? (or in other words are there infinitely many primes p so that p + 2 is also a prime.) No one knows! Students were informed that they had now entered the realm of research mathematics. They were working on something called the Twin-Prime conjecture, something mathematicians have been working on for hundreds of years [1, pg. 19]. Mathematicians know that finding twin primes is difficult. They are relatively scarce. But they also think that there may be infinitely many pairs of these primes that differ by two, but no one knows for sure.

The conclusion of the lesson occurred by stressing to the students the importance of the role of prime numbers. These numbers are the fundamental building blocks from which all numbers are built. The primes are what make the Fundamental Theorem of Arithmetic work.

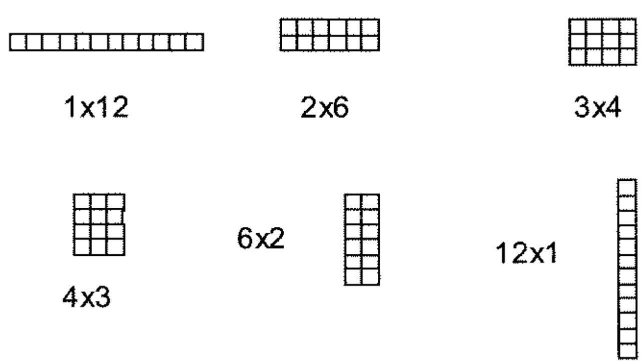
Lesson Extension

A possible extension to this activity is to have students explore numbers with a more complicated prime factorization. This would completely justify the chart used in class for the factors of the numbers up to 50.

From the previous examples of p^2, p^3, p^4 and p^5, the class could be challenged to speculate what the number of factors would be for a number with a prime factorization of $n = p^{k-1}$, where p is a prime number and k is a positive integer. They should be able to hypothesize that a number n with that type of factorization would have k factors. These factors would be 1, p, p^2, p^3,..., p^{k-1}.

The class could continue its exploration of this topic by considering numbers with a prime factorization of $n = p^2*q$ where p, q are both prime numbers. Students could first look at a relatively small number with that type of factorization. For example, the number $12=2^2*3$ could be used. The students could be asked to look at the rectangles built that form 12 and the list of the factors of 12, which are 1, 2, 3, 4, 6, 12 (Seen in figure 5). Students could rewrite these factors and group them in the following manner: 1, 2, $4= 2^2$; and 1(3), 2(3), 4(3). Students could be told that the second group is just the first group multiplied by three.

Figure 5



The class could also consider the example $45=3^2*5$. Similar to the past example, the factors of 45 turn out to be 1, 3, $9= 3^2$; and 1(5), 3(5), 9(5). Thus the number 45 has 6 factors. This process would work for any number n that has the prime factorization of p^2*q. The number n would have six factors, which are 1, p, p^2; and 1(q), p*(q), $p^2*(q)$.

Students could be asked to explore another process for finding the number of factors of any number n given its prime factorization. By adding 1 to the exponents appearing in each prime factorization, and multiplying the resulting numbers, the total number of factors is obtained. For example, $n = 36 = 2^2 \cdot 3^2$, since 4 has 3 factors (1, 2, 2^2) and 9 has three factors (1, 3, 3^2) then 36 has 9 factors. (Note that $(2+1)(2+1) = 9$. This is just the product of the number of factors of 4 multiplied by the number of factors of 9.) The students could be asked to use this procedure to verify that 12 and 45 each has 6 factors. Another example of this procedure could be given by $n = 30 = 2 \cdot 3 \cdot 5$ and since each prime 2, 3 and 5 has only an exponent of one (i.e. They each have only two factors 1 and the prime number itself.), the factors for the number 30 would be $(1+1)\cdot(1+1)\cdot(1+1) = 8$. This procedure would work for any number.

By including this portion of the lesson an instructor would be able to justify how the chart for the number of factors up to 50 (or for any number n) is built.

Conclusion

This lesson was designed for a class of seventh graders. But the authors feel that it could be adapted for use in a remedial mathematics course or even in an elementary mathematics methods course. It is an example of a lesson that gives the students the opportunity to use concrete, hands on manipulatives to explore an abstract aspect of mathematics. The students have the opportunity to connect geometric thinking with algebraic reasoning. In this lesson they are able to visually explore the Fundamental Theorem of Arithmetic, and get a taste of mathematical research and questions that still have no answer. Most importantly, it is an example of a lesson that shows the importance of creating visual connections with topics in mathematics.

References

1. Richard K. Guy, Unsolved Problems in Number Theory, Vol. 1, Second Edition, Springer-Verlag, 1991.