Modeling Wave Propagation through Metamaterials

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Introduction to Metamaterials

- My research dealt with materials recently discovered called metamaterials.
- Specifically metamaterials with a negative index of refraction.
- I studied ways to model wave propagation through them efficiently.
- I had to learn much of the physics behind them as well as derive the math.
What are Metamaterials?

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Metamaterials have several properties that make them interesting to study:

- Negative index of refraction is not found in nature.
- They derive this ability from their physical structure as opposed to chemical structure.
- Because of above they are usually cheaper to make.
In 1967 Victor Veselago proposed there could be man made materials that displayed an negative refractive index.

The refractive index $n$ is related to the permitivity $\mu$ and permeability $\epsilon$ by $n = \sqrt{\mu} \sqrt{\epsilon}$.

For more than 30 years, Vesalgo’s work was left as these kind of materials couldn’t be found.
Metamaterials, materials with a negative refractive index, were discovered by John Pendry in the 1990s.

Dr. Pendry discovered it was the physical structure of the material that caused its abilities.

Building on Veselago’s work, Pendry theorized that there existed a material with a negative refractive index. Working with others he was able to make the first metamaterial, the split ring resonator.

Figure: An array of split ring resonators - en.wikipedia.org
Applications of Metamaterials

Metamaterials have a wide range of options
- Make more efficient antennas.
- Make items invisible to sight and/or sound.
- Make good receivers.
- Used to make superlens or absorbers.
As with any material dealing with waves, we want to be able to model the waves as they pass through the material. There are several methods which are used now that can also be used with metamaterials. Finite difference time domain (FDTD) methods are very useful in solving partial difference equations in the time domain. FDTD is exactly what its name suggests, replace every differential with a finite difference and stay in the time domain.
There are two laws in Maxwell’s equations that relate magnetism and electricity: Faraday’s law and Amphere’s law.

For the theoretical construction of FDTD the magnetic fields and electric fields are staggered in time and space from each other.

Computationally this just means that you need to update one of the fields before the other.

It is also better to do an explicit calculation than an implicit calculation.

Figure: The grid of FDTD - fdtd.wikispaces.com
We can derive the one dimensional case for one of laws of Maxwell’s equation.

We know that for Faraday’s law in a one dimensional case that

\[-\mu \frac{\partial H_y(x,t)}{\partial t} = -\frac{\partial E(x,t)}{\partial x}\]

Replace every differential with a corresponding finite difference staggering the magnetic and electric fields.

You get

\[-\mu \frac{H_y(x+\frac{1}{2},t+\frac{1}{2}) - H_y(x+\frac{1}{2},t-\frac{1}{2})}{\Delta t} = -\frac{E_z(x+1,t)-E_z(x,t)}{\Delta x}\]

The only unknown is $H_y(x + \frac{1}{2}, t + \frac{1}{2})$. 

Derivation of FDTD
Resolving the boundary condition

- Since FDTD is discrete, we need a finite domain.
- If we set $E = 0$ and $H = 0$ then we get reflection.
- We would rather an absorbing boundary condition or an ABC.
- In 1994 Berenger offered an absorbing boundary condition that perfectly matched Maxwell’s equations.
- It was a perturbation to Maxwell’s equation assuming you are working with a homogeneous material.
Perfectly Matched Layer

- The perfectly matched layer (PML) is very useful for an ABC.
- The PML generates an actual layer that perfectly matches Maxwell’s equations except for a damping constant.
- Theoretically there is no reflection.

Figure: Basic idea of PML
Derivation of PML

- There is a damping coefficient $\sigma_x$ added to the wave equation.
- Where ever you find a $\frac{\partial}{\partial x}$ you replace it with a

$$\frac{1}{1 + i \frac{\sigma_x(x)}{\omega}} \frac{\partial}{\partial x}$$

- When $x$ is in the region you care about $\sigma_x(x) = 0$
- When $x$ is in the PML then usually $\sigma_x(x) = \alpha$ for some $\alpha > 0$
Problems with discretization

- The power of the PML is that you can make $\alpha$ as big as you want and the area of the PML as small as you want.
- However once you discretize it, it is no longer *perfectly* matching.
- The higher the $\alpha$ the more reflection.
- With an $\alpha$ of 2 and a large PML boundary, there is very little reflection.
My implementation of PML and FDTD with metamaterials, 1 Dimensional

Figure: FDTD with PML
After Berenger proposed the PML for Maxwell’s equation, quickly many derived something similar for other wave equations. It was shown to be equivalent to a coordinate transform to complex plane. Recently there has been those to formulate it into cylindrical and spherical coordinates and also try to solve it for nonhomogenous materials.
Further work

- Recently work has been done to derive the PML for cylindrical and spherical coordinates.
- However deriving the PML for non-homogeneous problems (problems with non-constant coefficients) is not clear on how to do it.
- With better formulation of the PML, the FDTD and PML will better model metamaterials.
- Simulation of metamaterials will allow better design.
- Test multiple metamaterial structures together