

**FROM THE SIXTY-FOURTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION**

Problem A1

Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: $4, 2 + 2, 1 + 1, 1 + 1 + 1 + 1$.

Problem A2

Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be nonnegative real numbers. Show that

$$(a_2 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}.$$

Problem A3

Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x .

Problem A4

Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that

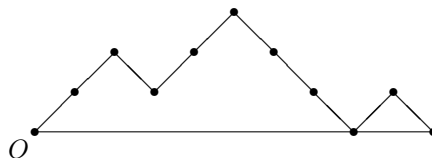
$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers x . Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|$$

Problem A5

A Dyck n -path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x -axis. A return is a maximal sequence of contiguous downsteps that terminates on the x -axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n -paths with no return of even length and the Dyck $(n - 1)$ -paths.

Problem A6

For a set S of nonnegative integers, let $r_s(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S, s_2 \in S, s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?

Problem B1

Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Problem B2

Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$, form a new sequence of $n - 1$ entries $\frac{3}{4}, \frac{5}{12}, \dots, \frac{2n - 1}{2n(n - 1)}$, by taking the averages of two

consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n - 2$ entries and continue until the final sequence produced consists of a single number x_n . Show that $x_n < \frac{2}{n}$.

Problem B3

Show that for each positive integer n ,

$$n! = \prod_{i=1}^n \text{lcm}\{1, 2, \dots, [n/i]\}.$$

(Here lcm denotes the least common multiple, and $[x]$ denotes the greatest integer $\leq x$.)

Problem B4

Let $f(z) = az^4 + bz^2 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then r_1r_2 is a rational number.

Problem B5

Let A , B and C be equidistant points on the circumference of a circle of unit radius centered at O , and let P be any point in the circle's interior. Let a , b , c be the distances from P to A , B , C respectively. Show that there is a triangle with side lengths a , b , c , and that the area of this triangle depends only on the distance from P to O .

Problem B6

Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx.$$