Problem A1

Let $n$ be a fixed positive integer. How many ways are there to write $n$ as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with $k$ an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: $4, 2 + 2, 1 + 1, 1 + 1 + 1 + 1$.

Problem A2

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_2a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2)\cdots (a_n + b_n))^{1/n}.$$

Problem A3

Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers $x$.

Problem A4

Suppose that $a, b, c, A, B, C$ are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$$

for all real numbers $x$. Show that

$$|b^2 - 4ac| \leq |B^2 - 4AC|$$

Problem A5

A Dyck $n$-path is a lattice path of $n$ upsteps (1,1) and $n$ downsteps (1,-1) that starts at the origin $O$ and never dips below the $x$-axis. A return is a maximal sequence of contiguous downsteps that terminates on the $x$-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.

Show that there is a one-to-one correspondence between the Dyck $n$-paths with no return of even length and the Dyck $(n - 1)$-paths.

Problem A6

For a set $S$ of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs $(s_1, s_2)$ such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets $A$ and $B$ in such a way that $r_A(n) = r_B(n)$ for all $n$?
Problem B1
Do there exist polynomials \(a(x), b(x), c(y), d(y)\) such that
\[
1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)
\]
holds identically?

Problem B2
Let \(n\) be a positive integer. Starting with the sequence 1, \(\frac{1}{2}\), \(\frac{1}{3}\), \(\cdots\), \(\frac{1}{n}\), form a new sequence of \(n - 1\) entries \(\frac{3}{4}, \frac{5}{12}, \cdots, \frac{2n-1}{2n(n-1)}\), by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of \(n - 2\) entries and continue until the final sequence produced consists of a single number \(x_n\). Show that \(x_n < \frac{2}{n}\).

Problem B3
Show that for each positive integer \(n\),
\[
n! = \prod_{i=1}^{n} \text{lcm}\{1, 2, \ldots, [n/i]\}.
\]
(Here lcm denotes the least common multiple, and \([x]\) denotes the greatest integer \(\leq x\).)

Problem B4
Let \(f(z) = az^4 + bz^2 + cz + d\) where \(a, b, c, d\) are integers, \(a \neq 0\). Show that if \(r_1 + r_2\) is a rational number, and if \(r_1 + r_2 \neq r_3 + r_4\), then \(r_1r_2\) is a rational number.

Problem B5
Let \(A, B\) and \(C\) be equidistant points on the circumference of a circle of unit radius centered at \(O\), and let \(P\) be any point in the circle’s interior. Let \(a, b, c\) be the distances from \(P\) to \(A, B, C\) respectively. Show that there is a triangle with side lengths \(a, b, c\), and that the area of this triangle depends only on the distance from \(P\) to \(O\).

Problem B6
Let \(f(x)\) be a continuous real-valued function defined on the interval \([0, 1]\). Show that
\[
\int_0^1 \int_0^1 |f(x) + f(y)| dy dx \geq \int_0^1 |f(x)| dx.
\]