FINAL HOMEWORK

Here is an assortment of problems I hope you will find interesting. There are many others in McKean's book. For the final homework, please do four problems of your choice.

1. Use Itô's formula to calculate the moments of the Gaussian distribution, $\beta_k(t) = \mathbf{E}[B_t^k]$ by proving first the recursive formula:

$$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds$$

for $k \geq 2$.

2. For constant c, $\alpha_1, \ldots, \alpha_n$ and an n-dimensional Brownian Motion $B_t = (B_1(t), \ldots, B_n(t))$, let

$$X_t = \exp(ct + \sum_{j=1}^{n} \alpha_j B_j(t)).$$

Prove that

$$dX_t = (c + \frac{1}{2}\sum_{j=1}^{n} \alpha_j^2)X_t\,dt + X_t(\sum_{j=1}^{n} \alpha_j\,dB_j).$$

3. Prove that the following stochastic processes are martingales with respect to the filtration of the Brownian motion B_t:

a) $X_t = e^{\frac{1}{2}t}\cos B_t$.

b) $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t)$.

4. Solve the mean-reverting Ornstein-Uhlenbeck equation:

$$dX_t = (m - X_t)\,dt + \sigma\,dB_t,$$

where m and σ are real constants. For a constant initial condition $\hat{x}$ calculate the mean and the variance of X_t.

5. Solve the system of stochastic differential equations (stochastically forced vibrating string equation):

$$dX_1(t) = X_2(t)\,dt + \alpha\,dB_1(t)$$

$$dX_2(t) = X_1(t)\,dt + \beta\,dB_2(t),$$

with a constant initial condition.

6. Let $x > 0$ be a constant and let

$$X_t = (x^{\frac{1}{3}} + \frac{1}{3}B_t)^3; \quad t \geq 0.$$

Prove that

$$dX_t = \frac{1}{3}X_t^{\frac{1}{3}}\,dt + X_t^{\frac{2}{3}}\,dB_t; \quad X_0 = x.$$

7. Consider the stochastic differential equation

$$dX_t = f(X_t)\,dt + e(X_t)\,db_t; \quad X_0 = x.$$

Assume that e does not vanish and let g be a C^2 function, satisfying

$$f(x)g'(x) + \frac{1}{2}e^2(x)g"(x) = 0.$$

Let (a,b) be an open interval such that $x \in (a,b)$. Define

$$\tau = \inf\{t > 0;\ X_t \notin (a,b)\}$$

and

$$p = \mathbf{P}[X_\tau = b].$$

Prove that

$$p = \frac{g(x) - g(a)}{g(b) - g(a)}.$$

Find a formula for p when

$$X_t = x + ct + \sigma b_t.$$

8. Consider Brownian motion with a drift:

$$X_t = b_t - at,$$

where $a > 0$. Find the distribution of the random variable

$$Y = \max\{X_t : t \geq 0\}.$$

HAVE FUN AND GOOD LUCK!