MIDTERM HOMEWORK

Below we use the definition of a stopping time used in newer literature: a random time τ is a stopping time of a filtration $(\mathcal{F}_t)$, $t \geq 0$ if for any t the event $\{\tau \leq t\}$ is in $\mathcal{F}_t$. A random time τ is an optional time of $(\mathcal{F}_t)$ if for any t the event $\{\tau < t\}$ is in $\mathcal{F}_t$.

0. As a warmup show that τ is an optional time of $(\mathcal{F}_t)$ if and only if it is a stopping time of $(\mathcal{F}_{t+})$, where

$$\mathcal{F}_{t+} = \bigcap_{u>t} \mathcal{F}_u.$$

1. Hitting times.

a) Let ξ_t, $t \geq 0$ be a stochastic process with values in $\mathbf{R}^d$ (or in any metric space) whose paths are continuous. Let Γ be a closed set. Prove that the the random variable $\tau_\Gamma = \min\{t : \xi_t \in \Gamma\}$ (by definition, $\tau_\Gamma = \infty$ if $\xi_t \notin \Gamma$ for any t) is a stopping time with respect to the filtration $\mathcal{F}_t$ associated with the process: $\mathcal{F}_t = \sigma(\xi_s : s \leq t)$.

b) Show an example where the analogous statement is no longer true when Γ is an open set.

c) Prove that if Γ is open, the hitting time τ_Γ is a stopping time with respect to the filtration $\mathcal{F}_{t+}$ (hence, by problem 0, an optional time of $(\mathcal{F}_t)$.

2. Background on stopping and optional times

a) Prove that if σ and τ are stopping times of the filtration $(\mathcal{F}_t$, then so are $\sigma \wedge \tau$, $\sigma \vee \tau$ and $\sigma + \tau$.

b) Let τ be a stopping time of $\mathcal{F}_t$. We define an event A to be measurable at time τ if for every $t \geq 0$ $A \cap \{\tau \leq t\} \in \mathcal{F}_t$. Prove that the collection $\mathcal{F}_\tau$ of all such events is a σ-algebra and that the random variable τ is measurable with respect to this σ-algebra. Show that when $\tau \equiv t$ is constant, then $\mathcal{F}_\tau = \mathcal{F}_t$.

c) Let τ be an optional time of $\mathcal{F}_t$ (stopping time in McKean's terminology, unfortunately!). McKean defines the σ-algebra $\mathcal{F}_{\tau+}$ as the collection of all events A such that $A \cap \{\tau < t\} \in \mathcal{F}_t$ (note the strict inequality!). Prove that this is a σ-algebra and that τ is measurable with respect to it. This can be done either directly, as part b) above, or combining the result of part b) with problem 0. As a consequence, show that if $\mathcal{F}_t = \mathcal{B}_t$ is the Brownian filtration and τ is as above, then $b(\tau)$ is measurable with respect to $\mathcal{B}_{\tau+}$. Since τ may be equal to ∞, this requires defining $b(\infty)$—as any constant or, as McKean does, as ∞.

3. Strong Markov Property of Brownian Motion

In this series of exercises we are going to go over McKean's proof of the strong Markov property of Brownian motion, as sketched on pages 10-11 of the book. We assume that τ is an optional time of $\mathcal{B}_t$ (again, this means a stopping time of $\mathcal{B}_{t+}$) and we are proving that conditionally on the event $\{\tau < \infty\}$ the process

$$b^+(t) = b(\tau + t) - b(\tau)$$

is a Brownian motion, independent of $\mathcal{B}_{t+}$.

a) Let us first explain what it means to be a Brownian motion, conditioned on an event. A simpler situation is: let C be an event with nonzero probability and Y—a random variable. To say that Y conditioned on C has distribution μ means simply that for any Borel set B in $\mathbf{R}$

$$\frac{P[C \cap \{Y \in B\}]}{P[C]} = \mu[B].$$

Note that for this to make sense, Y only needs to be defined on C. If E is another event, we can make a more detailed statement that, conditionally on C, Y is a random variable independent of E with the distribution μ. The statement becomes now

$$P[E \cap C \cap \{Y \in B\}] = P[C \cap E]\, \mu[B].$$

So, likewise, to say that a process defined on the event $\{\tau < \infty\}$ is—conditionally on this event—a Brownian motion, means that its finite-dimensional distributions satisfy

$$P[\{\tau < \infty\} \cap \{b^+(t_j) \in B_j, j = 1, \dots d\}] = P[\tau < \infty]\, \mu_W[x(t_j) \in B_j, j = 1, \dots d],$$

where μ_W is the Wiener measure on functions $x(t)$. Argue that, including in addition independence of $(\mathcal{B}_{t+})$, this is equivalent to the statement that McKean actually proves: with $e(t) = f(b(t+t_1) - b(t), \dots, b(t+t_d) - b(t))$, the claim is that for every $B \in \mathcal{B}_{t_+}\}$

$$E[e(\tau); B \cap \{\tau < \infty\}] = E[e(0)] P[B \cap \{\tau < \infty\}].$$

(In essence, we are replacing indicator functions by bounded continuous functions—an often employed device.)

b) With this setup and explanations, you should now be able to complete the proof. The main technical idea is to approximate τ by stopping times with dyadic rational values and use the assumption on B to factor the expectation in the last line on page 10.

c) Blumenthal's law: do Problem 1 on page 11.

d) As an application of Blumenthal's law, show that with probability one the Brownian path returns to 0 infinitely many times on any time interval $[0, \epsilon]$. Note: this is proven in McKean by a different method (problem 2, p. 19).

3. The zero set of Brownian motion

Define the zero set of the Brownian path by:

$$Z = \{t : b_t = 0\}$$

Part d) of the last problem proves that with probability one Z is infinite. Here we study Z in more detail.

a) Prove that with probability one the Lebesgue measure of Z equals zero. Hint: use Fubini theorem.

b) Use strong Markov property to show that with probability one Z does not contain an open interval.

c) Prove that with probability one Z has no isolated points. It is thus a closed set, dense in itself. Such sets are called perfect. The Cantor set is a well-known example of a perfect set.

d)* Prove that with probability one $Z \cap [0, 1]$ is homeomorphic to the Cantor set.

4. Some stochastic differential equations and Itô calculus.

a) Show that the process $X_t = \frac{b_t}{1+t}$ solves the stochastic differential equation

$$dX_t = -\frac{1}{1+t} X_t \, dt + \frac{1}{1+t} \, db_t.$$

b) "Brownian motion on an ellipse": let $X_1(t) = a \cos b_t$ and $X_2(t) = b \sin b_t$, with $a, b > 0$. Show that $X_1(t)$ and $X_2(t)$ satisfy the system of SDE's

$$dX_1(t) = -\frac{1}{2} X_1(t) \, dt - \frac{a}{b} X_2(t) \, db_t$$

$$dX_2(t) = -\frac{1}{2} X_2(t) \, dt + \frac{b}{a} X_1(t) \, db_t.$$

c) Solve the equations:

$$dX_t = X_t \, dt + db_t$$

and

$$dX_t = -X_t \, dt + e^{-t} \, db_t$$

with an arbitrary constant initial condition $X_0 = c$.

d) Let $a, b \in \mathbf{R}$. Consider the SDE

$$dY_t = \frac{b - Y_t}{1-t} \, dt + db_t; \quad 0 \le t < 1; \, Y_0 = a.$$

Verify that the process

$$Y_t = a(1-t) + bt + (1-t) \int_0^t \frac{db_s}{1-s}; \quad 0 \le t < 1$$

is a solution of the above equation, satisfying $\lim_{t \to 1} Y_t = b$ with probability one. This is one of the ways of defining an important Gaussian process—the Brownian bridge connecting a to b. Find the mean and the covariance of Y_t for $a = b = 0$ and verify that whenever B_t is a Brownian motion, the process $X_t = B_t - tB_1, \quad 0 \le t < 1$ has the same distribution as Y_t. The latter is a more standard way of introducing the Brownian bridge.

e) Let $(b_1(t), b_2(t))$ be a two-dimensional Brownian motion. The complex-valued process $b_t = b_1(t) + ib_2(t)$ is called a complex Brownian motion. Let F be an entire analytic function. For $Z_t = F(b_t)$ prove that $dZ_t = F'(b_t) \, db_t$. It follows that Z_t is a (complex-valued) martingale.

f) Following is an important Itô formula for an iterated stochastic integral:

$$n! \int_0^t db_{u_n} \Big(\int_0^{u_n} db_{u_{n-1}} \Big(\dots \int_0^{u_2} db_{u_1} \Big) \dots \Big) = t^{nover2} h_n\Big(\frac{b_t}{\sqrt{t}}\Big),$$

where

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}})$$

is the Hermite polynomial of degree n.

Verify the formula for $n = 0, 1, 2, 3$.

*Prove it for all n. Hint: first recall or look up the generating function of the Hermite polynomials. Note: there is an alternative approach to this problem in McKean's book (the top equation on page 38).

HAVE FUN AND GOOD LUCK!