Theorem If p is a prime number which has remainder 1 when divided by 4, then p can be written as a sum of two squares.

This theorem is hard to prove. We can attempt to check it by listing the primes whose remainder is 1 when divided by 4 (5, 13, 17, 29, 37, 41, 53, ...) and seeing if they are sums of two squares, but this does not constitute a proof.

$$
\begin{aligned}
5 &= 1+4 \\
13 &= 9+4 \\
17 &= 16+1 \\
29 &= 25+4 \\
37 &= 36+1 \\
41 &= 25+16 \\
53 &= 49+4
\end{aligned}
$$

Theorem If n is an integer, then $n^2 + n$ is even.

Greg's response: I believe it because if n is even, then n^2 is even and n is even, and even plus even is even.

If n is odd, n^2 is odd because odd times odd is odd, and your adding it to an odd, so $n^2 + n$ is even.

Emily's response: Because $n^2 + n = n(n + 1)$, and one of n and $n + 1$ is even and the other odd, and odd times even is even.

Some criticisms from the class: haven't proved the rules about multiplying odd and even numbers

Fix this by actually setting $n = 2k$, and substituting into $n^2 + n$, gives us $4k^2 + 2k = 2(2k^2 + k)$.

Proof the class came up with after much discussion:

First we need to define even and odd.

Definition: We define an integer n to be even if there exists an integer k such that $n = 2k$. We define an integer n to be odd if there's an integer k such that $n = 2k + 1$.

Proof of theorem: Write $n^2 + n = n(n + 1)$. It is well-known that every integer is either even or odd. There are two cases, $n = 2m$ or $n = 2m + 1$. For case 1,

$$n(n + 1) = 2m(2m + 1)$$

We claim that $2m(2m + 1)$ is even, because if $k = m(2m + 1)$, then by definition $2k$ is even and $2k = 2m(2m + 1)$. For case 2,

$$n(n + 1) = (2m + 1)(2m + 2) = 2(2m + 1)(m + 1)$$

This is also even, using $k = (2m + 1)(m + 1)$. Thus it's true.