1 Problem Analysis

The problem Start out with 99% water. Some of the water evaporates, end up with 98% water. How much of the water evaporates?

Guesses Solution: Guesses: Not 1%. 2%. 5%. Not 100%. 3%. 1.01%. 1.5%. 0.01%.

First solution We start out with 99 grams of water, 1 gram of stuff. After evaporation, there's 98 grams of water and 1 gram of stuff. So 1 gram of the water evaporated, which is a percentage of 1/99=1.01%.

Oops, that's wrong because $98/99 = .9898\ldots$, so there isn't 98% water left there's 98.98%.

But the approach is right, let's try again. We start out with 99 grams of water, 1 gram of stuff. After evaporation, we have 49 grams of water and 1 gram of stuff (because $49/50 = 98\%$). So 50 grams evaporated. That's $50/99 = 50.5\%$.

Algebraic analysis of the problem Replace 98 and 99 with letters. Start out with a being the initial fraction and b the final fraction of water.

Set up an equation to find out the fraction, x, of water that evaporates.

In the problem we just solved, $a = .99$, $b = .98$, and $x = .505$.

Suppose total is 1. Before evaporation, we have a amount of water and $1 - a$ amount of stuff.

Then xa evaporates, so $a - xa$ water is left. And $1 - a$ stuff is left. So the fraction of water left is

$$\frac{\text{water}}{\text{water} + \text{stuff}} = \frac{a - xa}{(a - xa) + (1 - a)} = \frac{a - xa}{1 - xa}.$$

This is supposed to be b, so we have

$$\frac{a - xa}{1 - xa} = b.$$

Solving this for x, we get

$$\begin{aligned} a - xa &= b - bxa \\ a - b &= xa - bxa \\ x &= \frac{a - b}{a(1 - b)}. \end{aligned}$$

Notice that in the original problem, we had $a - b = 0.01$ and $1 - b = 0.02$, and a very close to 1, so this fraction is about one-half.

2 Analogies between addition and multiplication

Both operations have an identity: do the operation with the identity on any number and you get the number back.

for addition it's 0:

$$x + 0 = x$$

for multiplication it's 1:

$$x \cdot 1 = x$$

Along with identities, we get inverses: do the operation with a number and its inverse, and get the identity. E.g.:

$$\begin{aligned} x + (-x) &= 0 \\ x \cdot x^{-1} &= 1 \end{aligned}$$

Powers and Multiples

What's a power? The nth power of x, which we write as x^n, is gotten by multiplying x by itself n times.

The nth multiple of x, which we write as nx, is gotten by adding x to itself n times.

So x^1 is the analog of $1 \cdot x$. And x^{-1} is the analog of $(-1)x$.

Here's an example of how this analogy works. We know that $0 \cdot x = 0$. What's the multiplication analogy of this statement?

Well, $0 \cdot x$ is the zeroth multiple of x, and the zero on the right is the additive identity. The analog of the zeroth multiple is the zeroth power, x^0, and the multiplicative identity. So maybe the analog we want is $x^0 = 1$.

Let's try and say this in a way that doesn't mention whether the operation is addition or multiplication. Let's call multiples and powers gerts. So the nth gert of a number is what you get by taking n xs and operating them together.

The rule we discovered on the previous page says that the zeroth gert of a number has to be the identity for the operation. Why is this so?

To understand this, let's think a bit about gerts in general. Suppose we take to gerts and perform the operation on them:

$$\text{second gert operate third gert} = \text{fifth gert}$$

$$x^2 \cdot x^3 = x^5$$

$$2x + 3x = 5x$$

So when you perform a zeroth gert you're not doing anything. So that's like operating with the identity. Nothing happens.

$$\begin{aligned} x + 0 \cdot x &= x \\ x \cdot x^0 &= x \end{aligned}$$

so that's why $0 \cdot x = 0$, and $x^0 = 1$.

3 Some definitions of function that we came up with.

- A function is a procedure that takes an input value and gives a single output value.
- A function is a correspondence between each element in a set A and a single element in a set B. (We call A the domain.)
- A function is a relationship between two separate quantities.
- A relationship between an independent variable and a dependent variable.

- A function is a set of ordered pairs satisfying the vertical line test (if (a, b) is on the graph, then there is no other point with first coordinate a on the graph) (in other words if (a_1, b_1) and (a_2, b_2) and if $a_1 = a_2$ then $b_1 = b_2$). (We didn't really come up with this one, the book did. But notice that it's really just saying in fancy language that "A function is the same thing as its graph.")

4 The role of definitions in mathematics

Is there a function from the empty set to $\mathbb{R}$?

Let's consider the definition: A function is a rule that assigns an element of set B to every element in set A. Let A be the empty set and B the real numbers.

The rule "assign the number 5 to every $x \in A$" defines a function from $\emptyset$ to $\mathbb{R}$.

Oops: we do not like this answer.

Try another definition. What about: for any sets A and B, a function $f : A \to B$ is a subset of $A \times B$ such that for every element $a \in A$ there is an element $b \in B$ such that (a, b) is in f.

If $A = \emptyset$ and $B = \mathbb{R}$, then $A \times B = \emptyset$. So we can define a function by specifying the empty subset of this cartesian product.

Using the rule definition there are infinitely many functions that map the empty set to $\mathbb{R}$.

Using the cartesian product there is one (the empty subset of $\emptyset \times \mathbb{R}$. (Mathematician-approved answer.)

5 Different ways of looking at functions

- Graph
- Table of values
- Operation
- Equation

Consider the function from $\mathbb{R} \setminus \{2\}$ to $\mathbb{R} \setminus \{0\}$:

$$f(x) = \frac{1}{x-2}$$

Let's look at the question of whether it has an inverse from the operational point of view. In terms of operations, we can think of this function as: subtract 2, take reciprocal.

To find the inverse function we reverse this: take reciprocal, add 2. A formula for this is:

$$f^{-1}(x) = \frac{1}{x} + 2$$

Check:

$$(f \circ f^{-1})(x) = \frac{1}{\frac{1}{x} + 2 - 2} = x$$

The conventional procedure for finding an inverse can be understood in terms of operations as well. The usual procedure is to write down a bunch of equations without words, something like:

$$\begin{aligned} y &= \frac{1}{x-2} \\ x &= \frac{1}{y-2} \\ \frac{1}{x} &= y-2 \\ y &= \frac{1}{x}+2. \end{aligned}$$

These equations don't really tell us what is going on. Why are we solving for y? Why do we swap the x and the y at the beginning? Here is a description of the process that puts the necessary words in between the equations.

Consider the function f given by

$$f(x) = \frac{1}{x-2}.$$

This function takes a number x and outputs the number y such that

$$y = \frac{1}{x-2}. \tag{1}$$

The inverse function, f^{-1}, is the function that takes y as input and outputs the corresponding number x. So, to get a formula for f^{-1}, we want to solve equation (1) for x. Now, it's conventional to name the input of a function x and the output y. So let's rephrase this as follows: The inverse function f^{-1} is the function that takes x as input and outputs the number y such that

$$x = \frac{1}{y-2}.$$

Now we solve this equation for y. To isolate y, we perform operations on both sides that "undo" the right hand side. Since the right hand side is obtained by subtracting 2 from y then taking reciprocals, we first undo the reciprocal by taking reciprocals of both sides:

$$\frac{1}{x} = y-2.$$

Then we undo the subtraction of 2 by adding 2 to both sides:

$$y = \frac{1}{x}+2.$$

Thus

$$f^{-1}(x) = \frac{1}{x}+2.$$

6 Operations

$$P(x,y) = 2x + 2y$$

What does it mean for this operation to be commutative?

$$\begin{aligned} P(x,y) &= P(y,x) \\ 2x + 2y &= 2y + 2x \end{aligned}$$

What does it mean for this operation to be associative?

$$\begin{aligned} P(P(x,y),z) &= P(x,P(y,z)) \\ 2P(x,y) + 2z &= 2x + 2P(y,z) \\ 2(2x+2y) + 2z &= 2x + 2(2y+2z) \\ 4x + 4y + 2z &= 2x + 4y + 4z \end{aligned}$$

Not associative.

7 Logarithms

The function $\log_{10}(x)$ is the inverse of the function 10^x.

So $\log_{10}$ "undoes" raising 10 to a power.

$$\begin{aligned} \log_{10}(10^x) &= x \\ 10^{\log_{10}(x)} &= x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= x \\ f(f^{-1}(x)) &= x \end{aligned}$$

with $f(x) = 10^x$ and $f^{-1}(x) = \log x$.

Or we could use e^x and $\ln x$.

$$\begin{aligned} 10^0 &= 1 \\ \log_{10} 1 &= 0 \end{aligned}$$

In general

$$\begin{aligned} 10^x &= y \\ \log_{10}(y) &= x \end{aligned}$$

$$\begin{aligned} 10^x 10^y &= 10^{x+y} \\ \log(AB) &= \log A + \log B \end{aligned}$$

8 Growth rates

Which is bigger, x^3 or x^2? On $[0,1]$, x^2 is bigger, and if $x>1$, x^3 is bigger.

Why? If $0<x<1$ then $x^3<x^2$ because $x^3=x\cdot x^2$ and $x\cdot x^2$ is x^2 multiplied by a number less than 1, and so it is smaller than x^2.

If $x>1$ then $x^3>x^2$ because $x^3=x\cdot x^2$ and $x\cdot x^2$ is x^2 multiplied by a number greater than 1 so it is bigger than x^2.

Which is bigger on the interval $x>1$, $0.001x^3$, or $1{,}000{,}000x^2$? Eventually, $0.001x^3$ is bigger.

Well, $0.001x^3=0.001x\cdot x^2$, so when $0.001x>1000000$ then that's when $0.001x^3$ becomes bigger. I.e., when $x>1000000000$.

In other words, to compare the two expressions, we look at their ratio:

$$\frac{0.001x^3}{1000000x^2}=\frac{x}{1000000000}.$$

When the numerator becomes larger than the denominator, the ratio is bigger than 1.

Which is bigger, $5x^3$ and $3x^3+1000000$? Look at ratio

$$\frac{5x^3}{3x^3+1000000}=\frac{5}{3+\frac{1000000}{x^3}}>\frac{5}{4}>1.$$

if $x^3>1000000$.

Which is bigger, x^3 or 2^x?

Set up the ratio

$$\frac{2^x}{x^3}$$

Lets look at what happens to this ratio as x goes from x to $x+1$. Compare

$$\frac{2^x}{x^3}\quad\text{with}\quad\frac{2^{x+1}}{(x+1)^3}$$

The second ratio is

$$\frac{2}{\frac{(x+1)^3}{x^3}}\approx 2$$

times bigger than the first if x is large. So the ratio between the two functions almost doubles every time you go from x to $x+1$. So if the ratio keeps doubling it eventually gets greater than 1. So that means 2^x is eventually bigger than x^3.

9 Problem Analysis

Maximize area enclosed by 40 ft of fence alongside a river.

Area A is given by

$$A=w(40-2w)=40w-2w^2,\quad \frac{dA}{dw}=40-4w$$

So maximum area occurs when $w=10$. Completely boring problem.

Replace 40 with L. What happens to the solution?

$$A=w(L-2w)=Lw-2w^2,\quad \frac{dA}{dw}=L-4w$$

So maximum area occurs when $w = L/4$.

Want to maximize

$$V = (L-2x)(1-2x)x = 4x^3 - 2(L+1)x^2 + Lx, \quad \frac{dV}{dx} = 12x^2 - 4(L+1)x + L.$$

Want to solve for x in the equation

$$12x^2 - 4(L+1)x + L = 0.$$

Solutions are

$$x = \frac{4(L+1) - \sqrt{16(L+1)^2 - 48L}}{24}$$

10 Equations

What is an equation?

$$\begin{aligned} 3x+6 &= 8 \\ 3 &= 1+2 \\ 3 &= 4 \\ (2x+1) &= x+(x+1) \end{aligned}$$

Am equation is a sentence of the form $a = b$ where a or b may algebraic expressions (including single numbers).

An equation by itself has no meaning; it is really more like a part of a sentence than a complete sentence. The same equation can fit into a number of different sentences with different meanings. For example:

- There exists a number x such that $3x + 6 = 8$.
- The number $x = 53$ is not a solution to the equation $3x + 6 = 8$.
- If $x = 2/3$ then $3x + 6 = 8$.
- For all numbers x we have $2x + 1 = x + (x + 1)$.

Notice that the last two have really different sorts of meanings: one is telling us about a particular number, the other about all numbers.

10.1 Solving equations

What is the definition of the solutions to $3x + 6 = 8$?

A procedural definition would tell us how to find the solutions: Subtract 6 from both sides, divide by 3.

$$\begin{aligned} 3x+6 &= 8 \\ 3x &= 2 \\ x &= \frac{2}{3} \end{aligned}$$

This is not really satisfying as a definition: it doesn't tell us what a solution is, just how to find one.

A nonprocedural definition: The set of solutions is the set of all numbers x for which $3x + 6 = 8$.

With this definition, the procedure can be expanded into a sequence of meaningful statements. For example:

We want to know if there is a number x such that when you multiply it by 3 and add 6, you get 8. If such an x exists, then

$$3x + 6 = 8.$$

If $3x + 6 = 8$ then

$$3x = 2,$$

because if two numbers are equal, then the two numbers we get by subtracting 6 from each one are also equal. Furthermore, if $3x = 2$ then

$$\frac{3x}{3} = \frac{2}{3},$$

because if two numbers are equal, then the numbers we get by dividing each one by 3 are also equal. Since $3x/3 = x$, this last equations tells us that

$$x = \frac{2}{3}.$$

10.2 Identities

Consider the identity

$$2x + 1 = x + (x + 1).$$

The expression on the left is equivalent to the expression on the right, because for every value of x the value of the expression on the left is equal to the value of the expression.

If two expressions are equivalent, the statement that they are equal is called an identity. An identity is an equation which is true for all values of the variables in them.

Superficially, proving an identity can look a lot like solving an equation. But the logic is different. This becomes clear if we write out all the words surrounding the equations, as we did with $3x + 6 = 8$.

For example, to prove the identity

$$2x + 1 = x + (x + 1),$$

we can write something like this:

Let x be any number. Then, since multiplying a number by 2 is the same as adding it to itself, we have

$$2x = x + x.$$

Now, if two numbers are equal, then the numbers I get by adding 1 to each one are also equal, so

$$2x + 1 = (x + x) + 1.$$

Furthermore, by the associative law,

$$(x + x) + 1 = x + (x + 1).$$

Now, if one number is equal to another, and if that number is equal to a third number, then the first number is equal to the third. So, since $2x+1=(x+x)+1$ and $(x+x)+1=x+(x+1)$, then

$$2x+1=x+(x+1).$$

Q.E.D.

Notice the difference between this argument and the one for the equation. In solving an equation, we start with the equation as a statement about a hypothetical number x, and try to deduce a statement of the form $x=$ number. In proving the identity, we started with any number x, and built up the identity using known properties of numbers.