Lesson 3: Equations

William McCallum

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Introduction
Last week we looked at a problem from a high school textbook, and saw that there was a lot more mathematics behind it than appeared at first sight. Now we will look at a high school subject, algebra. Students sometimes think there is nothing to algebra but techniques, but there are also some fundamental ideas in algebra, and thinking about the ideas will make you better at both doing and teaching algebra. The first idea, which we will spend the next two classes on, is the idea of an equation.

What is an equation?
Which of the following are equations?

\[ x^2 - 3x + 4 \]
\[ 3x + 6 = 8 \]
\[ 3 = 1 + 2 \]
\[ 3 = 4 \]
\[ (2x + 1) = x + (x + 1) \]

What is the definition of an equation?
An equation is a statement that two algebraic expressions are equal. Equations by themselves don’t mean much, they have to occur in sentences.

- There exists a number \( x \) such that \( 3x + 6 = 8 \).
- The number \( x = 53 \) is not a solution to the equation \( 3x + 6 = 8 \).
- If \( x = 2/3 \) then \( 3x + 6 = 8 \).
- For all numbers \( x \) we have \( 2x + 1 = x + (x + 1) \).

Solving equations
What is the definition of the set of solutions to \( 3x + 6 = 8 \)?
The solution set to \( 3x + 6 = 8 \) is \( \{2/3\} \).
To prove this, we need a definition of the solution set. The solution set to an equation is the set of all the values that will make the equation true.
Definitions are necessary if you want to be clear about what you claim. What exactly do I have to prove about a set of numbers in order to be able to claim that it is the set of solutions to an equation?

How do I prove that the solution set to $3x + 6 = 8$ is $\{2/3\}$?

I put $x = 2/3$ in the equation,

$$3 \cdot \frac{2}{3} + 6 = 8,$$

and check that the resulting statement is true.

We can easily check that $2/3$ is a solution to the equation. How do we know that we have found every solution? The only way to do this is write a mathematical proof. The statement we want to prove is that if $3x + 6 = 8$ then $x = 2/3$. How do we prove this?

Let $x$ be a number such that

$$3x + 6 = 8.$$

Then we want to show that $x = 2/3$. Since

$$3x + 6 = 8$$

then

$$3x = 2,$$

because if two numbers are equal, then the results of subtracting 6 from them are also equal, and so

$$x = 2/3,$$

because if two numbers are equal, then the results of multiplying them both by $1/3$ are also equal.

**Identities and Equivalent Expressions**

One of the equations we looked at was different from the rest. What are the solutions to $2x + 1 = x + (x + 1)$?

For all numbers $x$

$$2x + 1 = (x + x) + 1 = x + (x + 1).$$

The solution set of this equation is all real numbers.

The expression on the left is equivalent to the expression on the right, because for every value of $x$ the value of the expression on the left is equal to the value of the expression.

If two expressions are equivalent, the statement that they are equal is called an identity. An identity is an equation which is true for all values of the variables in them.

**Reasoning about solving**

Without solving it, what can we say about the solutions to the equation? Can you tell if there is a solution at all? What is the key part to look at to
decide this? Can you tell if 0 is a solution? Can you tell if the solution is negative?

\[ 3(x - (5 - 2x) + 42) = 2 - 6x \]

I know that this going to end up looking like

\[ ax + b = cx + d, \quad \text{with } a \neq c \]

so we will get

\[ ax = b, \quad a \neq 0 \]

so we will be able to solve by dividing by \( a \).

Do a similar analysis of the following equations.

\[ -6 - 8x = -2(3 + 4x) \]

\[ \frac{3x + 1}{4x + 1} = 1 \]

\[ \frac{4x - 1}{2x - 2} = 2. \]