1. Describe the zeros, vertical asymptotes and end behavior of the following functions. Explain your answers in terms of the algebraic structure of expressions for the functions (including, if necessary, transforming the expression into a different form).

   (a) $f(x) = \dfrac{e^x + 1}{e^{-x} - 1}$ **Answer**: This is a sample of a complete answer. The rest of the answers are in summary form. Since $e^x > 0$ for all x, the numerator is never zero, so there are no zeros. The denominator is 0 when $x = 0$. Since e^x is always increasing, this is the only place that it is zero, and it is small and positive just to the right of zero, and small and negative just to the left. So there is a vertical asymptote at $x = 0$. As x gets large, e^{-x} gets small, so the denominator approaches -1, and e^x grows without bound. So $f(x) \to \infty$ as $x \to \infty$. As $x \to -\infty$, the $e^x \to 0$, and $e^{-x} \to \infty$, so the denominator of the function grows without bound and the numerator approaches 1. Therefore $f(x) \to 0$ as $x \to -\infty$.

   (b) $g(x) = \dfrac{2}{x^2 + 1} - 1$ **Answer**: For $g(x)$ to be zero, $x^2 + 1$ must equal 2, so $x = \pm 1$ (or, put over a common denominator and factor). There are no vertical asymptotes since $x^2 + 1$ is always positive, and $g(x) \to -1$ as $x \to \pm\infty$.

   (c) $h(x) = \dfrac{1}{x - 1} - \dfrac{x}{x + 1}$ **Answer**: Writing this in the form

   $$h(x) = \frac{-(x^2 - 2x - 1)}{(x^2 - 1)}$$

   we see it has zeros at $x = 1 \pm \sqrt{2}$, vertical asymptotes at $x = \pm 1$, and a horizontal asymptote at $y = -1$.

   (d) $F(x) = \dfrac{\sin x}{x}$ **Answer**: The numerator is zero when $x = n\pi$, where n is an integer, and the denominator is zero when $x = 0$. So the zeros of $f(x) = 0$ when $x = n\pi$, n a nonzero integer, and at $x = 0$ there is neither a zero nor a vertical asymptote (in fct $g(x) \to 1$ as $x \to 0$. As $x \to \pm\infty$, $F(x) \to 0$, since $-1 \le \sin x \le 1$ for all x.

   (e) $G(x) = \dfrac{x^2 - 2x + 1}{(x^2 - 1)^2}$ **Answer**: No zeros, since the denominator is zero whenever the numerator is (namely at $x = 1$). At $x = -1$ there is a vertical asymptote. To see what happens at $x = 1$, factor the numerator and denominator and then cancel common factors to get out $G(x) = 1/(x + 1)^2$ for $x \neq 1$. So there is no vertical asymptote at $x = 1$, in fact $G(x) \to 1/2$ as $x \to 1$. As $x \to \pm\infty$, $G(x) \to 0$, since the denominator grows faster than the numerator.

   (f) $H(x) = \dfrac{\sin(e^x)}{e^{\sin x}}$ **Answer**: The denominator is never zero, since $e^y > 0$ for all real numbers y. The numerator is zero whenever $e^x = n\pi$,

for n an integer. In fact, since e^x only takes on positive values, we need only consider $n > 0$, and then we zeros at have $x = \log \pi + \log n$, for n a positive integer. As $x \to \infty$, the numerator oscillates more and more rapidly, while the denominator oscillates between e and e^{-1} with period 2π, so overall there is no simple description of the end behavior, except that it does not tend to infinity or any horizontal asymptote. As $x \to -\infty$, the numerator tends to $\sin(0) = 0$, so $H(x) \to 0$ also.

2. An integer n is defined to be even if and only if there is an integer k such that $n = 2k$, and odd if and only if there is an integer k such that $n = 2k + 1$. Use this definition to prove or disprove the following statements:

   (a) The product of two odd numbers is odd. **Answer**: This is true. Given two odd numbers, n_1 and n_2, we can write, by definitio of odd, $n_1 = 2k_1 + 1$ and $n_2 = 2k_2 + 1$, for some integers k_1 and k_2. Thus

   $$n_1 n_2 = (2k_1+1)(2k_2+1) = 4k_1k_2+2k_1+2k_2+1 = 2(2k_1k-2+k_1+k_2)+1.$$

   So, if $k = 2k_1k_2 + k_1 + k_2$, then k is an integer and $n_1n_2 = 2k + 1$, so n_1n_2 is odd.

   (b) The sum of three odd numbers is odd. **Answer**: This is true. Given three odd numbers, n_1, n_2, and n_3, we can write $n_i = 2k_i + 1$ for $i = 1, 2, 3$, for some integers k_1, k_2, and k_3. So

   $$n_1+n_2+n_3 = (2k_1+1)+(2k_2+1)+(2k_3+1) = 2(k_1+k_2+k_3+1)+1.$$

   So, if $k = k_1+k_2+k_3+1$, then k is an integer and $n_1+n_2+n_3 = 2k+1$, and so is odd.

   (c) If the sum of two numbers is odd, then one of them must be even. **Answer**: This is true. In general, the statement "if p then q" is equivalent to its contrapositive "if not p then not q", which in this case reads "if neither of the numbers is even then their sum must be even". If neither of the numbers is even, then both must be odd, and in this case their sum is indeed even.

   (d) The difference between the squares of two consecutive numbers is odd. **Answer**: This is true. Take two consecutive numbers. If the first number is n, then the one after it is $n + 1$, so the difference of their squares is

   $$(n + 1)^2 - n^2 = 2n + 1,$$

   which is odd.

3. Find a matrix for the linear transformation T such that

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}, \quad T\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}.$$

Does this matrix represent an isometry? **Answer**: If the matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then we have

$$\begin{aligned} a + 2b &= 3 \\ c + 2d &= 4 \\ 3a + 4b &= 1 \\ 3c + 4d &= 2 \end{aligned}$$

The solution is $a = -5$, $b = 4$, $c = -6$, $d = 5$, so the matrix is

$$\begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix}$$

This is not an isometry. For example, if we take the points $(0, 0)$ and $(0, 1)$, which are a distance of 1 apart, they get mapped to $(0, 0)$ and $(4, 5)$, which are not a distance of 1 apart.

4. For each of the following matrices, say whether or not it represents a rotation about the origin. If it does, give the angle of the rotation. If it does not, explain why not.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Answer: In each case we have to decide whether the matrix is in the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So the diagonal entries of the matrix must be equal, and the off-diagonal entries must be negatives of each other. All the entries must be between -1 and 1 (inclusive). And finally, the $\sin\theta$ and $\cos\theta$ must be sine and cosine of the same angle: we can check this by squaring and adding them, and seeing if that gives 1. All the matrices satisfy this condition. For (b), it is convenient to draw a 3,4,5 triangle and notice that the angle between the 3 and the 5 sides has cosine $3/5 = 0.6$ and sine $4/5 = 0.8$. For (d) you have to remember that $\cos \pi/4 = 1/x$. The angles are

(a) 0 (b) $\cos^{-1}(3/5) \approx 53.13°$

(c) π radians (d) $-\pi/4$ radians

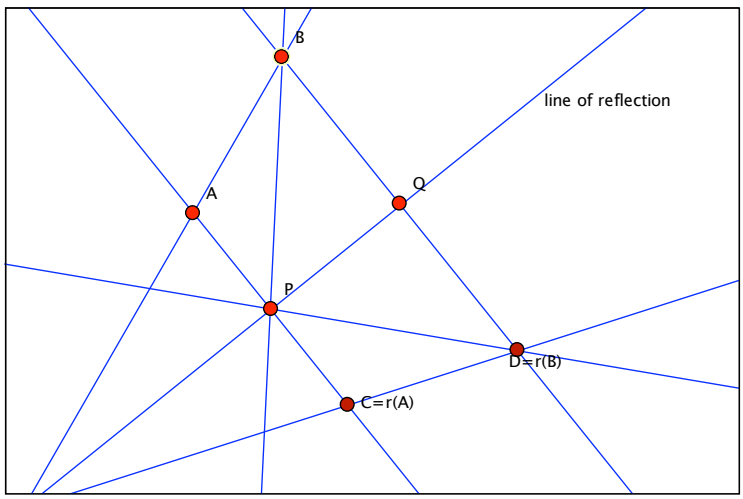
5. Calculate

$$\begin{pmatrix} \cos(\pi/37) & -\sin(\pi/37) \\ \sin(\pi/37) & \cos(\pi/37) \end{pmatrix}^{11}$$

Answer: Because the matrix represents a rotation through $\pi/37$ in the counterclockwise direction, multiplying it with itself 11 times represents a rotation of $11\pi/37$ in the counterclockwise direction. So

$$\begin{pmatrix} \cos(\pi/37) & -\sin(\pi/37) \\ \sin(\pi/37) & \cos(\pi/37) \end{pmatrix}^{11} = \begin{pmatrix} \cos(11\pi/37) & -\sin(11\pi/37) \\ \sin(11\pi/37) & \cos(11\pi/37) \end{pmatrix}^{11} =$$

6. Give a geometric proof that a reflection is an isometry. **Answer**:



Let r be the reflection about the line indicated. We want to show that, given any two points A and B, we have $AB = CD$, where $C = r(A)$ and $D = r(B)$. We consider the case where A and B are on the same side of the line; the case where they are on opposite sides is similar. By definition of reflection, we have $BQ = QD$, and $\angle BQP = \angle DQP$. Since $\triangle BQP$ and $\triangle DQP$ share the common side QP, they are congruent, by SAS congruence. Therefore $BP = DP$. Also, by definition of reflection, $AP = CP$, and $\angle APQ$ and $\angle CPQ$ are right angles. Thus, since $\angle BPQ = \angle DPQ$ (by the triangle congruence we proved earlier), we have $\angle APB = \angle CPD$. Hence $\triangle APB$ is congruent to $\triangle CPD$, so $AB = CD$, as required.

7. Prove or give a counterexample: every reflection is a linear transformation. **Answer**: This is not true. Any reflection about a line that does not pass through the origin constitutes a counter example, since in that case the point $(0, 0)$ will get moved to some other point, whereas a linear transformation has to take $(0, 0)$ to itself.

8. Give equations for a pair of lines l and m such that $r_l \circ r_m$ is a rotation of π radians about the point $(-1, 2)$. **Answer**: Any two lines at an angle of $\pi/2$ and both passing through $(-1, 2)$ will do. For example, $x = -1$, $y = 2$.

9. Let l be the line $y = -x$ and let m be the line $y = 1-x$. Give two different lines l' and m' so that $r_{l'} \circ r_{m'} = r_l \circ r_m$. **Answer**: Since these two lines are parallel, their composition is a translation in a direction from m to l, a distance equal to twice the distance between the two lines (the proof is similar to what we did in class for non parallel lines, where we got a translation). Any two lines parallel to these two, and the same distance apart, and so that the direction from m' to l' is the same as the direction from l to m, will do. For example, l' is the line $y = 2 - x$ and m' is the line $y = 3 - x$.

10. Say what sort of symmetry (translation, rotation, reflection, glide reflection, or no symmetry) the graph of f has in each of the following cases. For example, if $f(x) = f(-x)$, then the graph has reflection symmetry about the y-axis.

    (a) $f(x) = -f(-x)$ **Answer**: Rotation about the origin through π radians

    (b) $f(x + 2) = f(x)$ **Answer**: Translation by the vector $(2, 0)$

    (c) $f(x + 3) = -f(x)$ **Answer**: Glide reflection by the vector $(3, 0)$ and about the x-axis

    (d) $f(x - 3) = f(x) - 3$ **Answer**: Translation by the vector $(-3, -3)$

11. Is a rotation about the point $(1, 0)$ linear? Draw figures to illustrate your answer. **Answer**: No (unless the rotation is through an angle of 0, since then it is just the identity map). Rotation about the point $(1, 0)$ takes the origin to some other point, and a linear transformation has to take the origin to itself.