Review of Complex Numbers

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A complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. A remarkable fact about complex numbers is that you can add, subtract, multiply, and divide them, and still get a number of the same form. This is not at all obvious when it comes to division: how can you express

$$\frac{c + di}{c' + d'i}$$

in the form $a + bi$? (Answer this as an exercise.)

Another way of writing complex numbers is using the polar representation

$$r(\cos\theta + i \sin theta).$$

We call this the polar representation because if we represent the number $a + bi$ as the point (a, b) in the plane, then r and θ are the polar coordinates of the same point, so

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta.$$

Why should we bother to represent complex numbers this way? Well, another remarkable fact emerges when we try to multiply two complex numbers in polar form. If

$$z = r(\cos\theta + i\sin\theta) \quad \text{and} \quad z' = r'(\cos\theta' + i\sin\theta')$$

then

$$\begin{aligned} zz' &= rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') && (1)\\ &= rr'[(\cos\theta\cos\theta' - \sin\theta\sin\theta') + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')] && (2)\\ &= rr'[\cos(\theta + \theta') + i\sin(\theta + \theta')]. && (3) \end{aligned}$$

To get (3) we used the standard trig identities for sums of angles

$$\begin{aligned} \sin(\theta + \theta') &= \sin\theta\cos\theta' + \cos\theta\sin\theta' && (4)\\ \cos(\theta + \theta') &= \cos\theta\cos\theta' - \sin\theta\sin\theta'. && (5) \end{aligned}$$

Notice that (3) is in the form of a complex number whose absolute value is rr' and whose argument (angle with the positive real axis) is $\theta + \theta'$. So multiplication of complex numbers give us a wonderful new interpretation (and way of remembering) the identities (4) and (5): they are simply telling us that when you multiply two complex numbers you add their arguments.

This interpretation also gives a partial explanation of the identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{6}$$

since it tells us that $\cos\theta + i\sin\theta$ behaves like $e^{i\theta}$. When you multiply $e^{i\theta}$ and $e^{i\theta'}$ you also add θ and θ' because of the exponent laws. For a real explanation of (6) you need calculus or power series, as in the explanations that some of you found on the web.