What is an Equation?

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Introduction

High school students sometimes think there is nothing to algebra but techniques, but there are also some fundamental ideas in algebra, and thinking about the ideas will make you better at both doing and teaching algebra. The first idea, which we will spend the next two classes on, is the idea of an equation.

What is an equation?

Which of the following are equations?

$$x^2 - 3x + 4$$
 (1)

$$3x + 6 = 8 \tag{2}$$

$$3 = 1+2$$
 (3)

$$3 = 4$$
 (4)

$$(2x+1) = x + (x+1) \tag{5}$$

As you think about this, compare your response to different things in the list. For example, can (5) be an equation but (3) not be an equation? They are both true for all values of x. If you think (4) is not an equation, what do you think about 3 + x = 4 + x? Do we need equations to have solutions? If two equations are equivalent, can one of them be an equation and one not? Later I'm going to ask you to come up with a definition of equation, and thinking about this will help.

One way to view an equation is as a phrase that belongs in a bigger sentence. The sentence can be true or false; the same equation can fit into a number of different sentences in different ways. Consider the following list of sentences, for example.

- There exists a number x such that 3x + 6 = 8.
- The number x = 53 is not a solution to the equation 3x + 6 = 8.
- If x = 2/3 then 3x + 6 = 8.
- For all numbers x we have 2x + 1 = x + (x + 1).

Definition of Equation

Using the ideas so far, see if you can come up with your own definition of equation. Make sure your definition is consistent with the decision you made about which things were equations and which weren't? One of the things we have struggled with is whether and equation has to be a true statement or not. In the end we are all going to need to agree on a definition of equation, and we probably want it to be consistent with what most mathematicians see as the definition.

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Solving equations

Let's solve 3x + 6 = 8.

The solution to 3x + 6 = 8 is 2/3, and this is the only solution.

To prove this, we need a definition of the solutions. The solutions to an equation are all the values that make the equation true.

Definitions are necessary if you want to be clear about what you claim. What exactly do I have to prove about a set of numbers in order to be able to claim that it is the set of solutions to an equation? For example, how do I prove that the solution set to 3x + 6 = 8 is $\{2/3\}$?

We know that 2/3 is a solution, because when we put x = 2/3 in the equation,

$$3 \cdot \frac{2}{3} + 6 = 8,$$

we get a true statement.

How do we know that we have found every solution? The only way to do this is write a mathematical proof. The statement we want to prove is that if 3x + 6 = 8 then x = 2/3. How do we prove this?

Let x be a number such that

$$3x + 6 = 8.$$

Then we want to show that x = 2/3. Since

$$3x + 6 = 8$$

then

$$3x = 2$$

because if two numbers are equal, then the results of subtracting 6 from them are also equal, and so

$$x = 2/3$$

because if two numbers are equal, then the results of multiplying them both by 1/3 are also equal.

Preparation for next time: the rules of arithmetic

We will be talking more about the reasoning behind solving equations next time. In preparation I'd like you to think about the rules of arithmetic that you know. Please send them to the listserv. I'd like each student to contributed, either by proposing rules or responding to somebody else's proposal.