Reasoning in Algebra: Completing the Square

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Introduction to the five strands of mathematical proficiency

We have seen some examples of exercises that get students to look at algebraic expressions and think about the manipulations they are performing: to do mindful manipulation. Reasoning should infuse every mathematics lesson. As an example of this, we will consider completing the square, which is often taught as a mindless procedure. We will try to find ways of treating this topic that address the five strands of mathematical proficiency identified in Adding it Up

conceptual understanding—comprehension of mathematical concepts, operations, and relations

procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

strategic competence—ability to formulate, represent, and solve mathematical problems

adaptive reasoning—capacity for logical thought, reflection, explanation, and justification

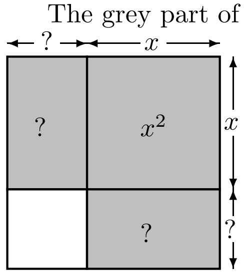
productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

For the rest of the class, we will try to think of ways of teaching completing the square, or exercises about it, that fall under each of these strands.

Conceptual understanding

Students sometimes use the "$b/2$ rule" when completing the square: to complete x^2+bx+c, write it as $(x+b/2)^2+c-b^2/4$. What are the "mathematical concepts, operations, and relations" behind completing the square? Can you think of a lesson or problem that gets at these?

The grey part of the following figure has area x^2+10x.



1. Fill in the question marks (the ones inside represent the areas of the rectangles containing them).
2. What is the area of the white square?
3. What is the are of the large square containing the entire figure?
4. If the area of the gray part is 39, what is the area of the total figure? Use your answer to deduce the value of x.

Strategic competence

One aspect of strategic competence is the ability to choose the form of an algebraic expression best suited to a given purpose. Students are often asked to factor and expand quadratic expressions, or to complete the square, but not often asked to think about which of these procedures they would choose in a given situation. This question is designed to test that ability.

A street vendor of t-shirts finds that if the price of a t-shirt is set at $\$p$, the profit from a week's sales is

$$(p-6)(900-15p).$$

Which form of this expression shows most clearly the maximum profit and the price that gives that maximum?

A. $(p-6)(900-15p)$

B. $-15(p-33)^2+10935$

C. $-15(p-6)(p-60)$

D. $-15p^2+990p-5400$

Adaptive reasoning

Adaptive reasoning includes the ability to build clear lines of argument out of known facts to establish new facts. For example, in explaining why B is the correct answer to the previous question, a reasoning student might provide the following:

> Because $(p-33)^2$ is a square, it is always positive or zero, and it is only zero when $p=33$. In the expression for the profit, a negative multiple of this square is added to 10,935. Thus the maximum profit is \$10,935, and the price which gives that profit is \$33.

Procedural fluency

Finally, procedural fluency plays an important role in supporting reasoning skills. For example, it is easy to get bogged down in the process of completing the square to derive the quadratic formula for solutions to

$$ax^2+bx+c=0.$$

A student with sufficient fluency in completing the square is more likely to be able to step back from the calculation and see the whole.

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx &= -c \\
x^2 + \left(\frac{b}{a}\right) x &= -\frac{c}{a} \\
x^2 + \left(\frac{b}{a}\right) x + \left(\frac{b}{2a}\right)^2 &= \frac{-c}{a} + \left(\frac{b}{2a}\right)^2 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$