Notes on Heron's Formula

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1 The Space of Triangles

Let a, b, and c be the side lengths of a triangle. We regard congruent triangles as being the same triangle. Or, more formally, we put an equivalence relation on the space of all triangles, where two triangles are equivalent if they are congruent. Then the equivalence classes are determined by the three parameters a, b, and c, since SSS congruence tells us that if two triangles have the same sidelengths they are congruent. The point (a, b, c) in 3-space corresponds to the congruence class of triangles that have side lengths a, b, and c.

Different choices of parameters can give the same congruence class, since we can label the sides of a triange in different ways. For example, the triples $(3, 4, 5)$ and $(3, 5, 4)$ both give rise to the congruence class of the right-angled triangle with sides 3, 4, and 5.

Also, not every triple corresponds to a triangle. For example, there is no triangle with sidelengths 1, 1, and 3, because the third side is too long, so the point $(1, 1, 3)$ does not correspond to a triangle. In the last class we saw that the set of points in 3-space corresponding to a triangle is

$$\{(a, b, c) : a > 0, b > 0, c > 0, a + b \geq c, b + c \geq a, c + a \geq b\},$$

which looks like a sort of 3-sided cone radiating out from the origin in the positive octant.

2 Area and Perimeter

I posed the question of whether two triangles with the same area and same perimeter were congruent. In view of what we figured out above, this ammounts to asking whether you can derive the three side lengths from the area and the perimeter. We have formulas that go the other way around, that is, that give us the area and perimeter in terms of the three side lengths:

$$\text{Perimeter} = P = a + b + c \tag{1}$$

and

$$\text{Area} = A = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \quad \text{(Heron's formula)}.$$

We can write Heron's formula is using the semiperimeter $s = (1/2)P$ (and while we are at it, let's square both sides to get rid of the square root):

$$A^2 = s(s-a)(s-b)(s-c). \tag{2}$$

From now on we will use s instead of P, so we replace equation (1) with

$$s = \frac{1}{2}(a+b+c). \tag{3}$$

So, our question becomes:

Given A and s, can you determine a, b, and c?

Last time we thought the answer was no, but we still don't have a counterexample. We start working with the following related question:

Given A, s, and c, can we determine a and b?

Equation (3) allows us to express b in terms of s, a, and c:

$$b = 2s - a - c. \tag{4}$$

Substituting this into equation (2) we get

$$A^2 = s(s-a)(a+c-s)(s-c). \tag{5}$$

Looking at equation (2) as an equation in a, with A, s, and c constant, we see that it is a quadratic equation. Since s and $s-c$ are constants, we divide through and get

$$\frac{A^2}{s(s-c)} = (s-a)(a+c-s).$$

Then we expand the right-hand side and collect terms that have the same power of a, getting

$$a^2 - (2s-c)a + \left(\frac{A^2}{s(s-c)} + s(s-c)\right) = 0 \tag{6}$$

and using the quadratic formula we get

$$a = \frac{1}{2}(2s-c) \pm \frac{1}{2}\sqrt{(2s-c)^2 - \frac{4A^2}{s(s-c)} - 4s(s-c)}. \tag{7}$$

So, equations (7) and (4) tell us how to express a and b in terms of A, s, and c, except that we need to know what values of these parameters yield real values for a, that is, give a non-negative value under the square root.

3 Work for Monday

1. Simplify the expression under the square root in equation (7).

2. Use equation (7) to find two non-congruent triangles with the same perimeter and the same area.

3. Use your answer to (2) to find an infinite set of triangles all of which have the same perimeter and same area.

4. Show that the radius of the incircle of a triangle is the same for all triangles with fixed area and perimeter. What is this radius for the triangles you found?

5. Notice that (4) means that all the triangles with a fixed area and perimeter can be visualized as circumscribed around a fixed circle. How does this help us find triangles with the same area and circumference?