The component group of a Néron model

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1 Introduction

In this paper we give bounds for both the exponent and the order of the group $\Phi$ of connected components of a Néron model of an abelian variety which has no multiplicative component in its reduction, and which has potentially good reduction. Both bounds depend only on the dimension $d$ of the abelian variety. The bound on the order is essentially due to Lenstra and Oort, who proved it in the case of purely additive reduction for the part of the order prime to the residue characteristic $p$. The omission of $p$ is remedied for almost all $p$ by our bound on the exponent. We have also generalized their argument slightly to include mixed good and additive reduction. The bound on the order is

$$
\sum_\ell (\ell - 1) \ord_\ell (|\Phi(\bar{k})|) \leq 2\lambda,
$$

if $p > 2\lambda + 1$,

where $\lambda$ is the dimension of the additive part of the reduction. The bound on the exponent $\epsilon = p^e_1 p_2^{e_2} \cdots p_r^{e_r}$ is

$$
\max\{p^{e-1}(p-1), \sum_{i=2}^r p_i^{e_i-1}(p-1)\} < 2\lambda
$$

with a minor adjustment for the prime 2.

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2 Statements of the theorems

If $F$ is a field we denote an algebraic closure of $F$ by $\bar{F}$, and a separable closure by $F^s$. Let $K$ be a henselian discretely valued field with finite residue field $k$ of characteristic $p$, and let $R$ be its ring of integers. We set $X = \text{spec}(R)$ and $x = \text{spec}(k)$. We denote the maximal unramified extension of $K$ by $K^u$. Let $A_K$ be an abelian variety of dimension $d$ over $K$ and let $A_X$ be its Néron model over $X$. Let $A_0$ be the connected component of $A_X$, and let $\Phi = A_x / A_0$. Then $\Phi$ is a finite étale group scheme over $k$, and $A_0$ is an extension of an abelian variety $B$ by a linear group, which decomposes over $\bar{k}$ as a product of a unipotent group $U$ and a torus $T$. Let $\alpha = \dim(B)$, $\lambda = \dim(U)$ and $\mu = \dim(T)$. Let

$$
\text{red} : A_K(K) \to A_\delta(k)
$$

be the reduction map and denote by $A^0(K)$ the subgroup of $A(K)$ which reduces to $A_\delta(k)$.

If $L$ is a finite extension of $K$ and $S$ the ring of integers in $L$, let $Y = \text{spec}(S)$, let $A_L = A_K \times_K L$, and let $A_Y$ be the Néron model of $A_L$. Let $y$ be the closed point of $Y$. We
say $A_K$ has good reduction over $L$ if $A_Y$ is an abelian scheme. From the defining property of the Néron model we get a map 

$$\rho : A_X \times_Y Y \to A_Y,$$

which induces a map 

$$\sigma : \Phi_1 \times y \to \Phi_y.$$ 

We denote by $\Gamma_y$ the kernel of this map.

**Theorem 2.1** Suppose that $\Phi_y(k) = \Phi_y(\bar{k})$. Let $L/K$ be a finite galois extension with galois group $G$. Then $\Gamma_y(\bar{k})$ is isomorphic to the kernel of the map 

$$H^1(L/K, A(L)) \to H^1(L^u/K^u, A(L^u))$$

(inflation to $H^1(L^u/K, A(L^u))$ followed by restriction).

**Corollary 2.2** Suppose that $A$ has good reduction over $L$ and let $i$ be the exponent of the inertia group of $L/K$. Then the group $\Phi_1(K)$ is killed by $i$.

Using the methods of Serre and Tate [?] we deduce the following corollary. If $n$ is a non-zero integer with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ we set 

$$L(n) = \begin{cases} p_2 - 1 & \text{if } n = 2 p_2, p_2 \text{ odd} \\ \sum_{i=1}^r p_i^{e_i - 1} (p_i - 1) & \text{otherwise} \end{cases}$$

**Theorem 2.3** Suppose that $A$ has potentially good reduction. Let $\epsilon$ be the exponent of $\Phi(\bar{k})$. Let $\lambda$ be the dimension of the additive part of $A^0_1$. Write $\epsilon = p^k \epsilon'$, with $(\epsilon', p) = 1$. Then $\max\{L(\epsilon'), L(p^k)\} \leq 2\lambda$. In particular, $\epsilon$ is divisible only by primes less than or equal to $2\lambda + 1$.

Lenstra and Oort [?] and Silverman [?] have obtained results bounding the order of the prime-to-$p$ part of $\Phi(\bar{k})$. Lenstra has obtained a bound on the order of all of $\Phi(\bar{k})$, but only for jacobians. Lenstra and Oort’s result gives a stronger bound than Silverman’s, but only applies to abelian varieties with purely additive reduction; however, their method extends easily to the case where $A$ has no multiplicative part in its reduction. Both results suffer from the deficiency that they only deal with the prime-to-$p$ part of $\Phi(\bar{k})$, whereas Corollary 2.2 bounds the exponent of all of $\Phi(\bar{k})$. Combining Lenstra and Oort’s result (suitably generalized) with Corollary 2.2, we prove 

**Theorem 2.4** Suppose that $A$ has potentially good reduction, and let $\lambda$ be the dimension of the additive part of $A^0_1$. Suppose that $p > 2\lambda + 1$. Then 

$$\sum (\ell - 1)\text{ord}_\ell(|\Phi(\bar{k})|) \leq 2\lambda.$$

If $p \leq 2\lambda + 1$, the same inequality holds if $p$ is excluded from the sum.

Lorenzini’s work suggests that Theorem 2.4 should hold without restriction on $p$. 


3 Proofs of the theorems

Let $G = \text{Gal}(\overline{K}/K)$ and $I = \text{Gal}(\overline{K}/K^u)$.

**Proof of Theorem 2.1.** We have a commutative diagram

$$
\begin{array}{cccccc}
H^1(X, \mathcal{A}_X) & \longrightarrow & H^1(K^u/K, A(K^u)) & \longrightarrow & \Phi_1(\overline{K}) \\
\downarrow & & \downarrow & & \downarrow \sigma \\
H^1(Y, \mathcal{A}_Y) & \longrightarrow & H^1(L^u/L, A(L^u)) & \longrightarrow & \Phi_1(\overline{K})
\end{array}
$$

The groups on the left may be interpreted as either flat or étale cohomology groups; the two are the same since $\mathcal{A}_X$ and $\mathcal{A}_Y$ are smooth [?], III.3.9. The left hand isomorphisms may be obtained by recalling that $\mathcal{A}_X = j^* A_K$ on the étale site and then identifying the Leray spectral sequence

$$
0 \to H^1(X, j^* A_K) \to H^1(K, A_K) \to H^0(X, R^1 j_* A_K)
$$

with the inflation-restriction sequence

$$
0 \to H^1(K^u/K, A(K^u)) \to H^1(K, A_K) \to H^1(K^u, A_K^u)^{G/I}
$$

via the description of étale sheaves given in [?], II.3.10-15. The right hand horizontal isomorphisms are proved in [?], I.3.8. The first vertical arrow is pullback composed with $\rho$, the second is just pullback; it is obvious that the left hand square commutes since $\rho$ is the identity map on the generic fiber. It is clear from the definitions that the perimeter of the diagram commutes, hence the right hand square commutes. Thus $\tilde{\Gamma}_y(\overline{K})$ may be identified with the group of cocycles in $H^1(K, A_K)$ which are unramified and die on restriction to $L$.

**Proof of Corollary 2.2.** By replacing $K$ and $L$ with suitable unramified extensions, we may assume that $\Phi(k) = \Phi(\overline{K})$ and that $L/K$ is totally ramified. Hence $i$ kills $H^1(L/K, A(L))$.

**Proof of Theorem 2.3.** It follows from [?] that the inertia group of $L/K$ must be a finite subgroup of $\text{Gl}(2\lambda, \mathbb{Z}_\ell)$ for all primes $\ell \neq p$. Further, the prime-to-$p$ part of the inertia group is cyclic, since it is the tame inertia group. Thus the theorem follows from the following lemma.

**Lemma 3.1** There is an element of order $m$ in $\text{Gl}(n, \mathbb{Z}_\ell)$ for all but finitely many $\ell$ if and only if $L(m) \leq n$.

**Proof.** Let $\zeta_m$ be a primitive $m$-th root of unity. Choose a prime $\ell$ such that $\mathbb{Q}(\zeta_m) \odot \mathbb{Q}_\ell$ is a field. Let $M \in \text{Gl}(n, \mathbb{Z}_\ell)$ have order $m$. Let $m$ have prime factorization $m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. Choose $d_1, \ldots, d_r$, so that

$$
d_1 \left( \frac{m}{p_1^{e_1}} \right) + \cdots + d_r \left( \frac{m}{p_r^{e_r}} \right) = 1
$$

and set

$$
M_i = M \left( \frac{m}{p_i^{e_i}} \right).
$$
Thus $M_i$ has order $p_i^{r_i}$, $M = M_1 \cdots M_r$ and $M_i M_j = M_j M_i$ for all $i, j$. Let $V$ be a basis of common eigenvectors for the $M_i$, and for each $v \in V$ set

$$o(i, v) = \text{order in } \overline{\mathbb{Q}_\ell} \text{ of the eigenvalue of } M_i \text{ on } v$$

and

$$o(v) = \phi(\prod_i o(i, v)),$$

where $\phi$ is the Euler totient function. For $v \in V$ there are at least $o(v)$ distinct elements $v' \in V$ for which $o(i, v) = o(i, v')$ for all $i$ (this may be seen using the action of $\text{Gal}(\mathbb{Q}_\ell(\mu_m)/\mathbb{Q}_\ell)$). Let

$$r(v) = \{ i : o(i, v) < p_i^{r_i} \}.$$

Then if $v \in V$ there are at least $o(v)$ distinct elements of $V$ such that $r(v') = r(v)$. Further, since $M$ has order $m$, $\cap_{v \in V} r(v) = \emptyset$. Thus, if $\{ r(v_1), \ldots, r(v_k) \} = \{ r(v) : v \in V \}$, $r(v_i) \neq r(v_j)$ whenever $i \neq j$, then

$$d \geq o(v_1) + \ldots + o(v_k) \geq \sum_{j=1}^k \sum_{i \in r(v_j)} \phi(o(i, v)) \geq \sum_{i} \phi(p_i^{r_i}).$$

The prime on the summation signs indicates that any term of the form $\phi(2) = 1$, the point is that $\phi(2n) \neq \phi(2) + \phi(n)$ if $(n, 2) = 1$. Finally, to see the sufficiency of the condition, use the embedding $\mu_{pr} \in \text{Gl}(n, \mathbb{Z})$ obtained by representing $\mu_{pr}$ on $\mathbb{Z}[\mu_{pr}]$ to construct an element of each prime power, then put these together in blocks. ■

**Theorem 3.2 (Lenstra-Oort)** Let $A$ be an abelian variety over $K$, and suppose that the reduction of $A$ has no multiplicative part. Let $\lambda$ be the dimension of the linear part of $\mathcal{A}_1^0$. Then

$$\sum_{\ell \neq p} (\ell - 1) \text{ord}_\ell(\Phi(\overline{K})) \leq 2\lambda.$$

**Proof.** We give a sketch only, referring the reader to [2] for the basic ideas. Recall that $\mathcal{A}_1^0$ is an extension of an abelian variety $B$ by a unipotent group. Consider the sequence

$$0 \rightarrow A_0(K) \rightarrow A(K) \rightarrow \Phi(k) \rightarrow 0.$$n

Let $T_\ell(A)$ be the Tate module of $A$ and let $M = T_\ell(A)^I$. Then $M \simeq T_\ell(B) \simeq T_\ell(\mathcal{A}_1^0)$. Thus lifting torsion points induces an injection

$$\Phi(k) \rightarrow A[\ell^n](K^s)/M_n$$

for large $n$, where $M_n$ is the image of $M$ under the projection $T_\ell(A) \rightarrow A[\ell^n](K^s)$. Further, the image is contained in $(A[\ell^n](K^s)/M_n)^I$. The proof now proceeds as in [2], with $X_\ell$ replaced by $X_\ell/M$. ■

Theorem 2.4 now follows immediately from Theorem 3.2 and the fact that the exponent of $\Phi(\overline{K})$ is divisible only by primes less than or equal to $2\lambda + 1$. 

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4 Concluding Remarks

1. The bound on the exponent of $\Phi(\overline{k})$ is much stronger than the bound on the order; this suggests that $\Phi(\overline{k})$ tends not to be cyclic.

2. If $d = 1$, then the groups allowed by Theorem 2.4 are $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. All these arise in practice.

3. Let $p$ be an odd prime and let $K = \mathbb{Q}_p(\zeta_p)$, where $\zeta_p$ is a primitive $p$-th root of unity. The jacobian of the curve $y^p = x^a(1 - x)^b$ ($p \nmid ab(a + b)$) over $K$ has $\Phi = \mathbb{Z}/p\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^{p-1}$ and $d = (p - 1)/2$ [?]. Both cases achieve the bound in Theorem 2.4.