Commentary on Kaminski et al, *The Advantage of Abstract Examples in Learning Math*, Science, April 2008

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3 June 2008

1 An uncontrolled variable

My fundamental objection to the experiment is that I believe that **the two treatment groups were not given the same mathematical structure to work with**.

Consider the following two groups (as in the abstract mathematical structure known as a group):

- Group A has 3 elements. Two of them behave in the following symmetric way: adding each to itself yields the other, and adding the two of them in any order yields the third element of the group. The third element behaves differently: adding it to any element of the group, in any order, yields that element back again.
- Group B has 3 elements, 1, 2, and 3. You add them just the way you normally add these numbers, except that if the ordinary sum is bigger than 3 you subtract 3 from it to get the sum in the group.

Of course, the two groups are isomorphic. However, let me propose three hypotheses:

1. some students in each of the treatment groups tried to learn the operations by discerning a structure, rather than just memorizing the operations
2. among students who tried to discern structure, those asked to work with the symbols oval, diamond, and flag discerned something like structure A, whereas students given the concrete examples discerned something like structure B
3. seeing the isomorphism between the two structures A and B is more cognitively demanding than grasping the structures separately.

The first hypothesis seems reasonable in a large randomized trial, and the third is borne out by my personal experience in teaching abstract algebra, where I find that most students can quite easily start calculating in abstract groups, but some never really grasp the idea of isomorphism.

Let me explain the second hypothesis in more detail. The symbols oval, diamond, and flag are visually quite distinct, and have short names. Thus one can imagine students looking at the diagrams saying something like this to themselves:

> well, oval and oval gives diamond, and diamond and diamond gives oval, that's easy to remember, and then when you add them together you get flag. How does flag work? Oh, I see, flag basically leaves everything the same.

This line of thinking leads in the direction of structure A.

On the other hand, for students working with the pictures of measuring cups, thinking this way would be quite cumbersome:

> well, one-third-filled-measuring-cup and one-third-filled-measuring-cup gives two-thirds-filled-measuring -cup, and two-thirds-filled-measuring-cup and two-thirds-filled-measuring-cup gives one-third-filled-measuring-cup ...

Moreover, the symbols used for the various measuring cups are visually similar. The way to tell them apart is to notice the levels. Thus, both the verbal descriptions and the symbols themselves cry out to be reduced to mere numbers. It's hard to imagine a structure-seeking student not taking this simplifying step and identifying the measuring-cup symbols with the numbers 1, 2, and 3, and working with the operations as operations on those numbers. This leads in the direction of structure B. (I have used the measuring cups here, but the same comments apply to the other "concrete" examples used in the study; they all viewed more easily as numbers than as abstract symbols.)

Now, the fundamental problem with the experiment is that the dichotomy between structure A and structure B is an unacknowledged confounding variable, in addition to the abstract/concrete dichotomy the authors wish to study. Structure A is treated abstractly, and structure B is treated with concrete examples.

2 Bias in the transfer task

The transfer task in the experiment, observing the children's game, is in clear danger of being biased towards structure A. The game is played with three visually distinct pictures with no obvious numerical reference. Students who have grasped structure A are prepared to see the same structure in the children's game. They can look for two pictures that behave in the symmetrical way described above, and a third picture that behaves differently. On the other hand, students who have grasped structure B have more work to do. They have to find a way of labeling the pictures in the game with numbers so that there is a correspondence with addition modulo 3. There are 6 different ways of labelling. Previous experience with ordinary addition does not lead one to expect that 1 and 2 have the symmetrical relationship noted above, or that 3 behaves differently from 1 and 2 under addition modulo 3. The fact that in the learning phase students had the option of calculating sums using the algorithm of addition modulo 3 means they might have missed these features of the structure, despite doing well on the test of their learning. I suspect that discovering and using these hidden structural features of division modulo 3 is a cognitively more demanding task than simply recognizing the re-occurrence of structure A if you have seen it once before.

Without further experiments, you can't distinguish the concrete/abstract variable from the structure variable, and the alternate hypothesis that the latter has an effect seems very reasonable to me. One might object that my structure variable isn't really a variable at all, since there is really only one underlying structure here, the group of order 3. But this strikes me as a highly sophisticated observation, likely to be beyond the level of many of the students in this study.

3 Possible further experiments

I see no reason to suppose that structure A is inherently abstract or structure B is inherently concrete (whatever these terms mean). I suspect the structure A/structure B variable is independent from the abstract/concrete variable. In the experiment, Structure A is treated "abstractly", and structure B is treated with "concrete" examples. However, one can imagine other experimental treatments that would reverse this. An abstract treatment for structure B would involve simply teaching the rules of addition modulo 3 as described in my previous posting, working with the actual numbers 1, 2, and 3. It's a bit harder to imagine a concrete example approach to structure A but perhaps one could give students apples, oranges, and bananas, and explain that that they can exchange two apples for an orange, two oranges for an apple, an apple and an orange for a banana, and so on. This is a bit artificial—why would people trade this way?—but then the same objection applies to the "concrete" examples in the experiment (why would two-thirds of a pizza and two-thirds of a pizza give me one-third of a pizza?).

A key issue is whether the difference between structures A and B is more salient in the Kaminsky et al experiment than the equivalence. One could imagine repeating the experiment with a different transfer task that reflected structure B. For example, one could devise a game in which each of two children holds up one, two or three fingers, and then a third child holds up some fingers in response. A winning response is one that corresponds to addition modulo 3. Do we expect that with this transfer task they would get the same results in favor of abstract learning? If you want to capture the underlying equivalence you need to design an experiment that is neutral with respect to different but equivalent forms.