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COMPUTER ALGEBRA AND HUMAN ALGEBRA

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ABSTRACT. The existence of computer algebra systems, whether or not we use them in our classroom, requires us to think more carefully about the problems we give our students. In particular, in trying to separate out the purely computational aspects of student work, the parts that could be done by a computer, we can see more clearly the parts that require mathematical understanding. I will illustrate this distinction with some examples of problems and projects that make use of computer algebra systems.

1. Computer Algebra

About ten years ago, I found a final exam from the calculus course at a prestigious United States university, and gave it to the symbolic manipuation program Mathematica. Here is the exam:

(1) Evaluate $\displaystyle\int \frac{x^2}{x^2-3x+2}\,dx$.

(2) Evaluate $\displaystyle\int \frac{dx}{(9-x^2)^{3/2}}$.

(3) State whether the following integrals converge or diverge, and give your reasons:
   (a) $\displaystyle\int_2^\infty \frac{x^2+4x+4}{(\sqrt{x}-1)^3\sqrt{x^3-1}}\,dx$.
   (b) $\displaystyle\int_0^1 (1-x)^{-2/3}\,dx$.
   (c) $\displaystyle\int_0^{\pi/2} \tan x\,dx$.

(4) State whether the following series converge or diverge and justify your answers.
   (a) $\displaystyle\sum_{n=4}^\infty \frac{1}{n(\ln n)(\ln(\ln n))^2}$.
   (b) $\displaystyle\sum_{n=1}^\infty \frac{5\sqrt{n}+100}{2n^2\sqrt{n}+9\sqrt{n}}$.
   (c) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1}}{\sqrt{n}}$.

(5) (a) Find the Taylor series at $x=0$ (McLaurin series) of $f(x) = x\cos\sqrt{x}$.
   (b) Find the radius of convergence of $\displaystyle\sum_{n=0}^\infty \frac{n^2}{5^n}x^n$.

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(6) Write all solutions of $z^3 = 8i$ in polar and Cartesian form, simplified as much as possible.

(7) (a) Find $\lim_{x\to 0} \dfrac{\cos(x^3) - 1}{\sin(x^2) - x^2}$.

(b) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{\sin n\theta}{n!}$.

(8) Find all real solutions of the following:

(a) $y'' - 2y' + 5y = 0$.

(b) $y'' - 2y' + 5y = \cos x$.

(9) The region between $y = x^{1/3}$, the x axis, and the line $x = 1$ is revolved around (a) the x axis, (b) the y axis. Find the volume in each case.

(10) Find the arc length of $y = \cosh x$ (i.e., of $y = \dfrac{e^x + e^{-x}}{2}$) from $x = 0$ to $x = 1$.

Notice that all the problems on this exam are requests to perform a computation, something computers are quite good at. Mathematica did indeeddo well on the exam, although it only gave the answers and did not show any work, and it had to resort to numerical approximation in some cases; for example, it did not appear to try any convergence tests on the series. Nonetheless, the fact that it can do essentially the entire exam is quite shocking, and has an effect on attitudes both of instructors and students.

There is, however, an important difference between the reaction of instructors and the reaction of students when they encounter this fact. Instructors tend to react by forming attitudes to the technology. Some want to ban the technology from the classroom, so that students will learn to carry out the procedures by hand. Others look for more interesting ways to use technology in their teaching. Either way, there is a focus on the technology: how to use it, or how to get rid of it.

Students, however, are more inclined to experience a change in attitude to the mathematics itself when they discover that it can be done by computer. They are likely to wonder why they must learn to do it by hand, and are going to need different justifications from the ones that might have worked in previous times. The sorts of questions exhibited here are likely to seem quite pointless to them. Maybe not ten years ago, when Mathematica was an exotic program to most students. But these days there are much more friendly "homework helpers" available on the web, which will not only give you the answers, but show the work necessary to achieve the answers as well. For now such things are not easily accessible in exam rooms. But soon we will have to start confiscating mobile phones as students enter the exam if we don't want such access.

Sooner or later, we must face the effect of technology on student attitudes to mathematics. It is worth spending some time now thinking about how human algebra and computer algebra might co-exist. In order to do this, I suggest that we enter a frame of mind where it is not the technology that is foremost, either as an eagerly welcomed tool or as a despised abomination, but rather one where the mathematics is foremost, and we imagine what it might be like to explore mathematical ideas in an environment where symbolic computational power is available at our fingertips. What questions would we like to ask our students in such an environment?

2. Using the Power of the CAS

It seems pretty clear that purely computational questions are not appropriate for CAS. Good CAS questions need an extra element, in addition to computation. One possibility is to take advantage of the CAS to pose questions which would normally have been too intensely computational. Here is an example.

> A population, P, in a restricted environment may grow with time, t, according to the *logistic function*
>
> (1) $$P = \frac{L}{1 + Ce^{-kt}}$$
>
> where L is called the carrying capacity and L, C and k are positive constants.
>
> (1) Find $\lim_{t\to\infty} P$. Explain why L is called the carrying capacity.
> (2) Using a computer algebra system, show that the graph of P has an inflection point at $P = L/2$.

One might imagine that students will do the first part by hand, and use a CAS for the second part. Using a CAS, we find

(2) $$\frac{d^2P}{dt^2} = -\frac{LCk^2e^{-kt}(1 - Ce^{-kt})}{(1 + Ce^{-kt})^3}.$$

Thus, $d^2P/dt^2 = 0$ when

$$1 - Ce^{-kt} = 0$$

and there is an inflection point at $t = -\ln(1/C)/k$, yielding

$$P = \frac{L}{1 + Ce^{\ln(1/C)}} = \frac{L}{1 + C(1/C)} = \frac{L}{2}.$$

Notice that there is a more elegant way to see this, suggested by the close proximity of equations (1) and (2): the second derivative is zero when $1 - Ce^{-kt} = 0$, which implies that $1 + Ce^{-kt} = 2$. A hand computation would have separated the function from its second derivative by a considerable number of lines and a considerable amount of student exhaustion, but the CAS, by juxtaposing the two, opens up the possibility for this kind of insight.

In this sort of problem, computers are viewed as tools that enable us to give more interesting and difficult questions. This is certainly a possibility. But it seems unlikely that students will make such a neat division between simple computations, which they do by hand, and more complicated ones, which they do by computer. If computers do all the algebra, what is left for humans? Quite a lot, it turns out.

3. Approaching Simple Questions Differently

Consider the problem of computing

$$\int_a^b \sin(cx)\,dx.$$

One might give this problem just after teaching the Fundamental Theorem of Calculus, expecting students to find an antiderivative for $\sin(cx)$ and use it. But what if a student uses a CAS to get

$$\int_a^b \sin(cx)\,dx = \frac{\cos(ac)}{c} - \frac{\cos(bc)}{c}.$$

Is there any way we can profit from the situation mathematically? We might ask the student to compare

$$\int_a^b \sin(cx)\,dx = F(b) - F(a)$$

with the answer from the CAS, which suggests that

$$F(x) = -\frac{\cos(cx)}{c},$$

a conjecture that is confirmed by taking the antiderivative. The Fundamental Theorem of Calculus appears in this line of reasoning, but rather as a vehicle for reflection than a computational tool. Nonetheless, it seems reasonable to expect that this sort of problem can be useful in teaching students to find antiderivatives.

4. Exploration and Conjecture

Another sort of CAS problem asks students to use the CAS as a probe in the mathematical world.

(1) Use a computer algebra system to differentiate $(x+1)^x$ and $(\sin x)^x$.
(2) Conjecture a rule for differentiating $(f(x))^x$, where f is any differentiable function.
(3) Prove your conjecture by rewriting $(f(x))^x$ in the form $e^{h(x)}$.

Answers from different computer algebra systems may be in different forms. One form is:

$$\begin{aligned}\frac{d}{dx}(x+1)^x &= x(x+1)^{x-1} + (x+1)^x \ln(x+1)\\ \frac{d}{dx}(\sin x)^x &= x\cos x(\sin x)^{x-1} + (\sin x)^x \ln(\sin x)\end{aligned}$$

Both answers follow the general rule

$$\frac{d}{dx} f(x)^x = x f'(x)\,(f(x))^{x-1} + (f(x))^x \ln(f(x)),$$

which the student can then verify. This problem calls on skills of pattern recognition that are not required when students are asked to compute derivatives by hand. Indeed, the skill of logarithmic differentiation is often learned in a purely mechanical way by students, so that although they can carry it out, they do not always see the regularity and structure in the answers it produces. The CAS reverses the process of learning by asking students first to see the regularity and structure, and then to learn the procedure.

5. Puzzles

If you play around with a CAS for a while, you are bound to come across problems like the following, which are best described as mathematical puzzles.

(1) Use a computer algebra system to find $P_{10}(x)$ and $Q_{10}(x)$, the Taylor polynomials of degree 10 about $x = 0$ for $\sin^2 x$ and $\cos^2 x$.
(2) What similarities do you observe between the two polynomials? Explain your observation in terms of properties of sine and cosine.

The Taylor polynomials of degree 10 are

$$\begin{aligned} \text{For } \sin^2 x, \qquad P_{10}(x) &= x^2 - \frac{x^4}{3} + \frac{2\,x^6}{45} - \frac{x^8}{315} + \frac{2\,x^{10}}{14175} \\ \text{For } \cos^2 x, \qquad Q_{10}(x) &= 1 - x^2 + \frac{x^4}{3} - \frac{2\,x^6}{45} + \frac{x^8}{315} - \frac{2\,x^{10}}{14175} \end{aligned}$$

The coefficients in $P_{10}(x)$ are the opposites of the corresponding coefficients of $Q_{10}(x)$. The constant term of $P_{10}(x)$ is 0 and the constant term of $Q_{10}(x)$ is 1. Thus, $P_{10}(x)$ and $Q_{10}(x)$ satisfy

$$Q_{10}(x) = 1 - P_{10}(x).$$

This makes sense because $\cos^2 x$ and $\sin^2 x$ satisfy the identity

$$\cos^2 x = 1 - \sin^2 x.$$

In this problem the puzzle is to explain a surprising pattern in the coefficiens, and a simple identity is the key to unlocking the puzzle. This puzzle is not only enjoyable, but reinforces the connection between the Taylor polynomial and the corresponding function.

Let me conclude with one more extended puzzle.

For positive a, consider the family of functions

$$y = \arctan\left(\frac{\sqrt{x} + \sqrt{a}}{1 - \sqrt{ax}}\right), \qquad x > 0.$$

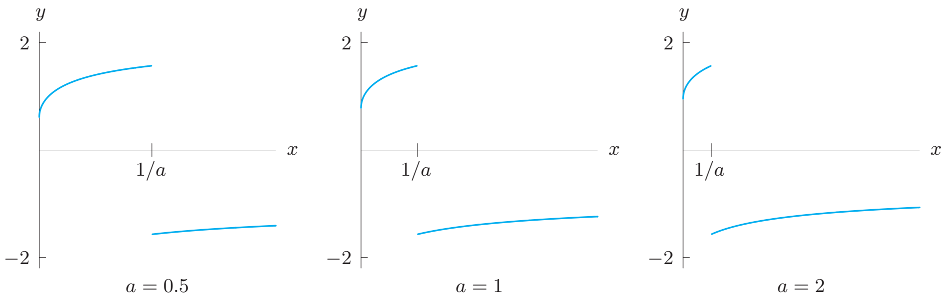
(1) Graph several curves in the family and describe how the graph changes as a varies.
(2) Use a computer algebra system to find dy/dx, and graph the derivative for several values of a. What do you notice?
(3) Do your observations in part (b) agree with the answer to part (a)? Explain.

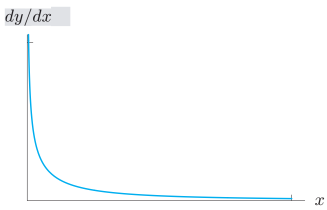
The graph has a jump discontinuity whose position depends on a. The function is increasing, and the slope at a given x-value seems to be the same for all values of a.

Most computer algebra systems will give a fairly complicated answer for the derivative. Here is one example; others may be different.

$$\frac{dy}{dx} = \frac{\sqrt{x} + \sqrt{a}\,\sqrt{a\,x}}{2\,x\,\left(1 + a + 2\,\sqrt{a}\,\sqrt{x} + x + a\,x - 2\,\sqrt{a\,x}\right)}.$$

When we graph the derivative, it appears that we get the same graph for all values of a.





How do we explain what is going on here? Since a and x are positive, we have $\sqrt{ax} = \sqrt{a}\sqrt{x}$ (a CAS usually needs to be told this fact, since it can't be assumed in general). We can use this to simplify the expression we found for the derivative:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\sqrt{x} + \sqrt{a}\sqrt{ax}}{2x\left(1 + a + 2\sqrt{a}\sqrt{x} + x + ax - 2\sqrt{ax}\right)} \\
&= \frac{\sqrt{x} + \sqrt{a}\sqrt{a}\sqrt{x}}{2x\left(1 + a + 2\sqrt{a}\sqrt{x} + x + ax - 2\sqrt{a}\sqrt{x}\right)} \\
&= \frac{\sqrt{x} + a\sqrt{x}}{2x\left(1 + a + x + ax\right)} = \frac{(1+a)\sqrt{x}}{2x(1+a)(1+x)} = \frac{\sqrt{x}}{2x(1+x)}
\end{aligned}
$$

Thus the derivative is independent of a. This explains why all the graphs look the same in part (b). (In fact, of course, they are not exactly the same, because $f'(x)$ is undefined where $f(x)$ has its jump discontinuity.)

6. Conclusion

There is a common thread to all these problems: they reverse the normal order of things. Whereas a paper-and-pencil problem ends with the answer, or at least with the result of a computation, a CAS problem starts with that result. In each case, there is a chance to contemplate the answer given by a CAS and see structure in it. This has the effect that the work the problem demands is more about reflecting on or interpreting a symbolic form, rather than in producing it. This is a radical change in the cognitive demand of algebra problems. Although we might in retrospect wish that we had all along been asking students to reflect on their answers, this piece is usually lost, and students in general do not think of algebraic expressions as objects for interpretation. Rather they think of them as grist for the computational mill; things to which something—anything—must be done. The CAS, by pulling the carpet out from under the skill of doing things, brings to the fore the fact that algebra is a subject with meaning as well as actions. In the computer age, human algebra is more a matter of reflection, interpretation, and pattern recognition, and less a matter of computation.

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