Thinking out of the Box

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Beyond the box metaphor

Students, parents, and teachers often think of technology as a black box that performs calculations in an unknown way, the results of which are available for more studied use. Some value the black box for its ability to skip mindless drudgery and get to the concepts; others fear that it replaces the mental exercise which is necessary preparation for deeper understanding. Some educators seek to avoid this problem, while still taking advantage of technology rather than banning it, by making the black box a white box, that is, a transparent device that shows each step in the calculation, although the steps themselves are carried out by machine. Whatever the merits of technology, the box metaphor itself has bad effects:

- ◇ Students treat the boxes, black or white, as oracles: "The answer must be right because my calculator said so". This permits them to avoid responsibility for their answers.
- ◇ Teachers treat them as miraculous time-saving devices: "I won't have to teach that because they'll do it with the calculator." This is a dubious assumption: although chain-saws are more powerful than axes, it takes just as long to learn how to use one.
- ◇ Critics of technology treat them as having strange, mind-sapping powers: "When the calculator turns on, the brain turns off". This deflects attention from other, less magical causes for mindless behavior, such as popular attitudes to the nature of mathematics and to intellectual endeavor in general.

Although there is a grain of truth in each of these beliefs, the metaphor of the black box amplifies them to an irrational degree, perhaps because of its resonance with the myth of Pandora's box, or with that other real but problematic box in our lives, the television, to which is attributed the same array of oracular, miraculous, and demonic abilities.

The undesirable consequence of accepting the black box metaphor is that the black box, not the mathematics, becomes the center of attention. Thinking of the technology as a white box, whose internal operations are visible, is a helpful first step towards refocusing on the mathematics, not the technology. However, I would like to suggest that for the purposes of considering its uses in teaching, we would do better to let technology out of its metaphorical box entirely, so that it can take its place in the general landscape beside the older miracles of paper and pencil. Indeed, the power of computer algebra systems is no longer confined to black boxes, but is freely available on the web [1]. Immense computational power is in the air we and our students breathe. In our daily lives we accept and adjust to new technological miracles; we welcome their advantages, recognize their dangers,

[1] Try, for example, `http://www.integrals.com`.

and ultimately adjust to them, forget about them, and continue with the important business of living. It's a good idea to do the same in our teaching lives.

Putting the mathematics first

In order to consider sensibly the uses of technology, we must, paradoxically, start out by forgetting about it. In the same way that we might plan a trip by first choosing a destination on a map, and then considering the details of getting there, we should start out by considering first our mathematical goals, and then consider the ways we and our students can get there. Furthermore, just we would, in planning a trip for our children, consider not only the means of travel we ourselves might choose but the ones they might prefer, we should, in planning how to teach our students, consider the new technologies they might use in addition to the old technologies of paper and pencil that we grew up with.

Thus, I propose the following procedure for thinking about how to use computer algebra systems in teaching:

Step 1. Think of a piece of mathematics you would like your students to learn: a calculation, concept, or connection.

Step 2. Formulate a question that tests this knowledge. Put yourself in the student's shoes and solve using the various technologies available (including both paper and pencil and computer algebra systems). Better still, ask a real student to have a go at the problem, if one is available.

Step 3. Evaluate the mathematical content of the solution, revise Step 1, and repeat.

No doubt this is unremarkable advice. However, it's worth making explicit because, even today, computer algebra systems are treated like the new kitchen gadget we just bought and don't quite know what to do with. We are still in those first few days when you keep trying to think of ways to use the gadget (or to avoid using it). The more sensible approach is to decide what you want to eat first, prepare it using what comes to hand, and see if you like the result. A willingness to try new technologies as well as old, coupled with good taste and common sense in evaluating the results, will ensure that the new gadget finds its niche.

I'd like to illustrate the process with a very simple dish. There are many examples of interesting, creative problems that use the full power of a CAS. However, if the CAS is only to be used in specific preparations—if it turns out to be the pasta machine of mathematics education rather than the food processor—then perhaps we don't need a whole book devoted to it. So I have chosen a fairly simple example to work on, more or less at random, and will perform four rounds of the process. Perhaps the most surprising result is the extent to which our undertaking in Step 2 to consider all the *technologies* available to our students forces us to focus on underlying *mathematical* issues.

Round One: The distinction between technique and insight

Topic for the day: Solving equations

We decide we want to see if our students can take an equation and put it in a form amenable to solution using the quadratic formula, so we formulate the following problem.

PROBLEM 1 *Solve the equations*

$$x + \frac{1}{x} = 0, \qquad x + \frac{1}{x} = 2, \qquad x + \frac{1}{x} = 4.$$

Here are the solutions using paper and pencil and using a CAS, with our marginal comments.

PAPER AND PENCIL SOLUTION.

Good problem!

(a) Tests ability to put equation into standard form.

(b) Tests ability to choose the appropriate method: direct observation, factoring, or the quadratic formula.

$$x + 1/x = 0$$
$$x^2 + 1 = 0$$
$$x = \pm\sqrt{-1}$$
$$\boxed{\text{No solution}}$$

$$x + 1/x = 2$$
$$x^2 + 1 = 2x$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$\boxed{x = 1}$$

$$x + 1/x = 4$$
$$x^2 + 1 = 4x$$
$$x^2 - 4x + 1 = 0$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 1}}{2}$$
$$\boxed{x = 2 + \sqrt{3} \text{ or } x = 2 - \sqrt{3}}$$

■

CAS SOLUTION.

What?

Requires barely any effort beyond pushing a few buttons; no evidence that the student understands anything. Ban the black box!

```
solve(x+1/x=0,x)
                         false
solve(x+1/x=2,x)
                         x = 1
solve(x+1/x=4,x)
                         x = -(√3-2) or x = √3 + 2
```

■

Before continuing, it is worth noting that the student's marginal comments might be quite different from ours. A natural reaction on the student's part would be to question why he or she is being required to learn paper and pencil solutions when the calculator does such a good job. One of our challenges is to think of some good responses to this question.

Returning to our problem, we worry after seeing the CAS answer that students who use CASs are learning nothing more than how to operate a machine. Granted that the machine replaces much of what we used to teach, but will students learn any judgement in using it? Will they know how to evaluate their solutions and recognize when they don't make sense? Perhaps most importantly, will they learn to recognize situations where they don't need to use it? For example, it is possible to see right away that $x + 1/x = 0$ cannot have a solution, because if x and $1/x$ are both defined then they are both non-zero and of the same sign.

Then it occurs to us that the same questions might be asked of the students who gave the paper and pencil answer. We look at it again with newfound concerns about what is going on inside the students' heads, and realize that, although it demonstrates the abilities (a) and (b) that we noted in the margin, it's possible that these are reflexive behaviors rather than instances of conscious judgement. Is it possible that the student in the first answer is functioning somewhat like the CAS in the second answer?

Back to the drawing board

Returning to Step 1 of the process, we think again about what it is we really want the students to know. We remember that we chose these equations because we expected students to recognize that they were quadratic equations in disguise. We want students to be able to look at an equation and know what moves can be made with it, just as a good chess player can look at a board and see potential traps and opportunities. For example, we would like students to be able to anticipate the degree of the transformed equation and thus predict the number of solutions. This leads us to formulate a different problem using the same equations:

PROBLEM 2 *For what values of a does*

$$x + \frac{1}{x} = a$$

have no solutions? Exactly one solution? More than one solution?

We'll pick up the solution to this problem in a moment, but first we pause to reflect.

What have we learned from Round One?

Consideration of the CAS answer to our problem has forced us to acknowledge as an explicit goal in our teaching what one might call *algebraic insight*: the ability to discern the structure of an algebraic expression and the potential forms inherent in it. This is a purely mathematical goal, not dependent on technology. However, before the CAS came on the scene, it was easy to suppose that it was a concomitant of technical skill, and therefore ignore it. (Possibly this was always a dubious supposition, but that's another story.)

Just as the CAS forces us to pay attention to algebraic insight, it also causes us to reconsider the role of paper and pencil algebraic technique. It may still be that training in algebraic technique is good preparation for the development of algebraic insight, but we can no longer offer to our students what was, for me as a child, a great motivator: the thought that, by learning to solve equations, I was learning the way into a universe that none who couldn't follow me could enter. The ability merely to carry out the steps in solving Problem 1 is no longer the only key to the magic kingdom. Now, apparently, there is another way in. Even those who think it a false and easy way have a responsibility to ensure that students who use it arrive in good shape, since we cannot stop students from using it any more than we can stop them from breathing.

Round Two: Brave new world

Now let us consider the solutions to Problem 2.

Paper and Pencil Solution.

$$\begin{aligned} x + 1/x &= a \\ x^2 + 1 &= ax \\ x^2 - ax + 1 &= 0 \\ -ax &= -x^2 - 1 \\ a &= \frac{-x^2-1}{-x} = \frac{x^2+1}{x} \end{aligned}$$

$$\boxed{??}$$

■

First CAS Solution.

```
solve(x+1/x=a,x)
```

$$\mathtt{x} = \frac{-\left(\sqrt{\mathtt{a}^2-4}-\mathtt{a}\right)}{2} \ \mathtt{or}\ \mathtt{x} = \frac{\sqrt{\mathtt{a}^2-4}+\mathtt{a}}{2}$$

If $a^2 > 4$ the expression under the square root sign is positive, so there are two roots. If $a^2 < 4$ it's negative and there are no roots. ■

Second CAS Solution.

```
solve(x+1/x=0,x)
                false
solve(x+1/x=2,x)
                x = 1
solve(x+1/x=4,x)
```

$$\mathtt{x = -(\sqrt{3}-2)\ or\ x = \sqrt{3} + 2}$$

There are no roots if $a < 2$, one if $a = 2$, and two if $a > 2$. ■

Third CAS Solution.

```
factor(x+1/x-a)
```

$$\frac{\mathtt{x}^2 - \mathtt{a}\cdot \mathtt{x} + 1}{\mathtt{x}}$$

Multiplying by x, we see that we must solve a quadratic equation. The quadratic formula gives

$$x = \frac{a \pm \sqrt{a^2-4}}{2}.$$

If $a > 2$ or $a < -2$, then the expression under the square root sign is positive, so there are two distinct roots. If $a = \pm 2$, it is zero, so there is just one root. And if $-2 < a < 2$, it is negative, so there are no roots. (Also, since $a^2 - 4 \neq a^2$, none of the roots is ever zero, so there are no extraneous roots resulting from multiplication by x). ■

At this stage we begin to regret our pious words about letting technology out of the box. What were we thinking? The paper and pencil student has been completely thrown by the introduction of the parameter a. Although this student can transform algebraic equations accurately, it seems that he doesn't know which variable to solve for.

The first CAS student is in better shape, probably more through good luck than anything else. Entering the same command as before, but with an a in place of the 0, 2, or 4, the student has been presented with the solution by the CAS. To her credit, she manages to do something with the output. There is a significant new step in this problem: recognizing for which values of a the square root $\sqrt{a^2 - 4}$ exists, and for which values it is zero. However, the solution illustrates a problem with the CAS. It has presented the two roots in a slightly non-standard form, which makes it more difficult to detect the case when they are equal, and the student has indeed missed that case. Also, her division into cases $a^2 > 4$ and $a^2 < 4$ is not entirely satisfactory—does she know exactly what ranges of a these inequalities define?

The second CAS student, not realizing that he can put the equation with the parameter into his CAS, has just re-entered the three equations from Problem 1 and made a guess based on them. He has forgotten that a could be negative, and has no real mathematical justification for his answer. On the other hand, he has shown something that neither the paper and pencil student nor the first CAS student demonstrated; an awareness that he is dealing with a family of equations parameterized by a, and must consider their solutions in x for different values of a.

As for the third CAS student, it's clear that one of her parents must be a math teacher. This kid is brilliant! Not only does she know the quadratic formula, not only does she realize the need to eliminate the possibility of a zero root, but she has discovered she can get the CAS to do her algebra for her by putting an equation in the form *something* $= 0$ and then factoring *something*. This shows exactly the sort of judgement we want our students to have: clever use of technology coupled with masterful insight.

We return once again to Step 1, this time wondering whether any of the students other than the last one have a solid grasp of what an equation is. So we formulate the next question to test that.

PROBLEM 3 *Are there three distinct numbers a, b, c such that*

$$a + \frac{1}{a} = b + \frac{1}{b} = c + \frac{1}{c}?$$

In order to answer this question, the student must recognize that it can be reformulated as follows: is there a number d such that the equation $x + 1/x = d$ has three distinct solutions? We are requiring the student both to supply the variable and introduce a new parameter. It will be very interesting to see how they handle this question!

What have we learned from Round Two?

First, we've learned that it's difficult to come up with good questions. We no longer know what sorts of answers, right or wrong, to expect. Students will use the CAS in ways we had not anticipated; they will miss the point we had intended in asking the question, but perhaps stumble on another point we had not thought of. We have to learn to expect a wider variety of answers, and be more careful in interpreting them.

Second, CAS syntax both helps and hinders the student. The syntax required by the `solve` command forces the student to state explicitly at the outset which variable he or she is solving for, and therefore prevents the CAS student from getting lost the way the paper and pencil student did. On the other hand, the CAS's output can sometimes be in an unanticipated form that makes the student's job harder.

Finally, we've learned that there's no going back. Even if the CAS disappeared tomorrow, we've just discovered something about the fragility of the paper and pencil student's understanding that we can't ignore. The paper and pencil exercises we have been giving were designed to build up into a complete picture; the student was meant to be able to put it all together and develop some independent thinking. Unfortunately, the moment we vary the wording of the question slightly, or introduce a seemingly harmless parameter, we derail the student entirely. The good news is that there is still plenty of work for teachers even with the arrival of the CAS. Contrary to our fears, it seems that the CAS doesn't provide the student with any magic way of avoiding thought. Indeed, the only student who has truly mastered the CAS is the one who clearly would have thrived without it. For the others, perhaps there will be as many learning opportunities using the CAS as without it. However, with or without, it's clear that we have to work harder at drawing those opportunities out.

Round Three: Back to Basics

We've been waiting eagerly for the answers to Problem 3. Disappointingly, all the pages are blank. No one knows what to make of this question. A pity, since it was such a good one. In the process of reformulating the original problem to take account of the original CAS solution, we have also made it much more difficult. What started out as a simple exercise problem has become a test of sophisticated and flexible algebraic knowledge.

Don't we still need problems like Problem 1 in order the build the intuition needed for Problems 2 and 3? Won't we have to insist those are done by hand? We return to Step 1 with a renewed interest in the connection between mechanical algebraic skills and algebraic insight. And here we find ourselves wandering into unknown territory. It's difficult for those of us brought up in the pre-CAS era to imagine how one might develop the ability to recognize that $x + 1/x = a$ can be transformed into a quadratic equation without having done many hours of paper and pencil exercises. We possess the insight because we can move the symbols rapidly in our heads as we have moved them so often on paper, performing a sort of fast-forward computation where we keep track only of certain aspects of the equation, such as its degree. I don't know if it's possible to develop these abilities without paper and pencil skills, but it seems safe to suppose that at least some paper and pencil drill will still be necessary.

By the same token, however, elementary exercises with the CAS are probably necessary if we want students to use it flexibly and thoughtfully later on. The answers we have seen so far suggest a number of possibilities. First, the clever use of the `factor` command, which our bright student discovered earlier on, should perhaps be added to the library of standard techniques that we teach our students. Second, we noticed that the CAS students were aided by the necessity of entering the variable to be solved for as an argument to the `solve` command. Perhaps we can exploit this feature of the CAS to develop exercises that will test if students know what to do when they have a choice of variables.

Here is a problem which aims to exercise both paper and pencil and CAS skills.

Problem 4 *Given a, for what values of x does the following equation hold?*

$$x^2 - ax + 1 = 0.$$

Given x, for what values of a does it hold?

Notice that I have avoided the instructions "Solve for x" and "Solve for a". It is quite possible for a student to be adept at performing the paper and pencil activity triggered by these two instructions, but to be unable to say which is being asked for when in Problem 4. Similarly, the student with the CAS is being challenged to enter the correct command: $\mathtt{solve}(\mathtt{x}^2 - \mathtt{ax} + 1 = 0, \mathtt{x})$ or $\mathtt{solve}(\mathtt{x}^2 - \mathtt{ax} + 1 = 0, \mathtt{a})$.

What have we learned from Round Three?

Most of all we have learned that there are lots of questions to which we do not know the answers. Computer algebra systems have made us think more about algebraic insight: the ability to recognize algebraic structure, an understanding of the meaning of equations, an appreciation of algebraic manipulations as steps in a process of reasoning. However, students must have practise with basic procedures to develop these higher skills. I haven't provided a solution to the problem in this round, because I think at this stage of our enquiry we have passed the limits of armchair speculation (if we didn't pass it a long time ago). We need to decide which of the traditional drill questions still serve a purpose in developing algebraic insight, and which were merely training our students to be good human algebra systems. The same goes for the CAS. We will have to develop drill exercises for the CAS as well that support the development of algebraic insight. Mathematics education research has something to say about this, and so does the practical everyday experience of teachers experimenting with CASs in their classrooms. It is important to recognize in the course of this experimentation, however, that our goals have fundamentally changed. Efficient and rapid computational ability is no longer an end itself, because if that's all we want we can get it from the technology. Rather, it is a means to a higher end, possibly many higher ends. We need to put more though into what are those higher ends, and develop more practical experience about which drill exercises support them.

Round Four: Getting thinking out of the black box

The ease with which students can solve equations on the CAS opens up another category of problems that was not feasible before. If we play with the CAS and find solutions to $x + 1/x = a$ for various values of a, we notice patterns:

```
solve(x+1/x=4,x)
```

$$x = -(\sqrt{3}-2) \text{ or } x = \sqrt{3} + 2$$

```
solve(x+1/x=5,x)
```

$$x = \frac{\sqrt{21}+5}{2} \text{ or } x = \frac{-(\sqrt{21}-5)}{2}$$

```
solve(x+1/x=6,x)
```

$$x = -(2 \cdot \sqrt{2} - 3) \text{ or } x = 2 \cdot \sqrt{2} + 3$$

The rational part of each root (that is, the summand not involving a square root) seems always to be $a/2$. This is a simple consequence of the quadratic formula. Can we turn it around: get the students to see the pattern, try to explain it, and ultimately derive the quadratic formula in this case? The following problem heads in this direction. It is more in the nature of an extended project in that it might require repeated attempts and guidance from the teacher.

PROBLEM 5 *Consider the equation*

$$x + \frac{1}{x} = a.$$

1. *Solve the equation for various different values of the constant* a*. Find the sum and the product of the roots, and describe any patterns you notice.*
2. *Suppose that* r *is a root of the equation. Explain why* $1/r$ *must also be a root. Use this fact to explain the patterns you noticed in part 1.*

Call the two roots of the equation r and s. By the end of this problem, the students have discovered a proof that $r + s = a$ and that $rs = 1$. There are a few possible directions one could take from here. One would be to make the connection with the related quadratic equation and its factored form:

$$x^2 - ax + 1 = (x - r)(x - s) = x^2 - (r + s)x + rs.$$

Alternatively, one could first generalize the project to the equation

$$x + \frac{b}{x} = a.$$

Playing around with the solutions to this equation using a CAS, and then repeating the steps of the problem, again leads to a natural conjecture and proof. In this case the related quadratic equation

$$x^2 - ax + b = 0$$

has a quite general (although nonstandard) form.

Finally, one could extend this project into a derivation of the quadratic formula, either for the special equation $x^2 - ax + 1 = 0$ or for the more general form $x^2 - ax + b = 0$. Having found a formula for the sum and product of the roots, one is led to the following question: if we know the sum and the product of two numbers, do we know the numbers themselves? Here again there is room for exploration with a CAS. If the CAS can solve simultaneous non-linear equations, then we can ask it to solve $x + y = a$, $xy = b$ for various values of a and b, and try to see patterns. This would make for an extended project that, combined with the previous one, would provide a proof of the quadratic formula.

What have we learned from Round Four?

The quadratic formula is an interesting example, because many argue that it is no longer necessary to teach it, because students can use calculators to find roots (either numerically or exactly). It is true that if one is only interested in finding roots then one might as well use the CAS. On the other hand, the derivation of the quadratic formula is a beautiful piece of mathematics, one that as often regarded as beyond the reach of the typical high school algebra class. Paradoxically, while the CAS buries the quadratic formula as algorithm, it resurrects the possibility of the quadratic formula as mathematical reasoning. This is because it replaces the procedure of assertion-followed-by-derivation with observation-followed-by-explanation. This makes an enormous difference from the student's point of view. For the student who has not acquired the algebraic insight to follow it, a derivation is a leap in the dark that may end up merely being a leap of faith rather than a leap of understanding. Lost in the middle of a derivation and heading towards a conclusion, the statement of which has only been half understood, a student may well conclude that mathematics is a meaningless game and that he or she is not a good player.

If, however, the student starts with the conclusion, obtained through direct observation, then the situation is quite different. It is not possible for the student to proceed to the step of explaining the observation until it has been clearly formulated, because it is the student that has the responsibility for making the formulation. This increases the chance that the student will understand the statement he or she is trying to derive. The process of explaining it is no longer a game, but an act of scientific enquiry. The difference is the same as the difference between studying the life of Sherlock Holmes and the life of Shakespeare, or between studying the anatomy of a dragon and the anatomy of a horse. Students use technology to remain engaged with mathematical reality, instead of making things up as they go along.

Conclusion

The existence of symbolic computational capacity outside the human brain is something fundamentally new in our world, which is likely to change the mathematical life of students in our classrooms in the same way that the automobile changed the physical life of an earlier generation. To ensure that change for the better outweighs change for the worse, we need to first accept this fundamentally new thing and become knowledgeable about its capabilities, but then relegate it to the background while we focus on our mathematical goals. One way of doing this, which I have sketched in some detail in this essay, is to perform thought experiments with specific mathematical problems and see how they work it in this new technology-saturated environment.

I'd like to conclude with three questions that have come out of this thought-experiment.

What is the role of mental exercise in developing mathematical ability?

It is commonly recognized that physical exercise is necessary for health; people with cars bicycle to work, and people who drive their cars to work make time in the day to engage in physical activity with no useful purpose other than the exercise it gives the body. By the same token, although many basic mathematical operations can now be performed faster and more accurately outside the human brain than inside, it seems likely that there will continue to be a need for the human brain to perform some of those operations as exercises, for the sake if its own mental health. The question is, what sort of exercises? More importantly, how do we present these exercises to our students and convince them of their benefits? The old standby, "You have to know how to do this or you won't be able to get on in life" is no longer convincing, because students see technology as a way of getting on in life without the mental operations it replaces.

What technological skills are necessary for developing mathematical ability?

Just as we want our students to be safe and attentive drivers, we want them to use technology thoughtfully. In order to find out what constitutes a thoughtful use of technology, we need to use it thoughtfully ourselves. By picking up a CAS and working through problems with it, we can begin to lay down rules of the road for using it. It is important to temporarily shut out the back-seat driver in our minds who keeps reminding us of the way all this used to be done, or could be done without the technology. The creative use of the `factor` command in the third CAS solution to Problem 2 is an example of what I have in mind as skillful driving here; it's the sort of thing we might think of when we give up trying to use CASs simply to mimic what used to go on in paper and pencil calculations, and accept them instead as new and interesting inhabitants of our mathematical environment.

Do CASs provide new ways of fostering mathematical reasoning?

Just as the automobile opened up new worlds of travel to our ancestors, so the CAS opens up new worlds of mathematics. It enables our students to conquer the mathematical Wild West of their imaginations, full of scary critters and dangerous people, by showing them the reality. It is up to us to invite them further to think about and explore what they see; to make sure that their voyages of the mind are like hikes through Yosemite, not visits to the mall.