Algebra as a Dynamic Environment

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The University of Arizona

Σ P Δ 2009

The Five Lessons of This Talk

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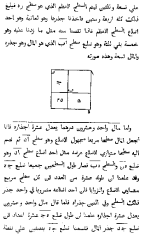
Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830



. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

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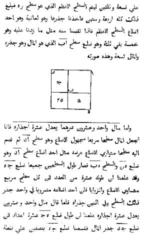

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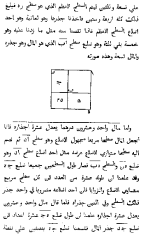

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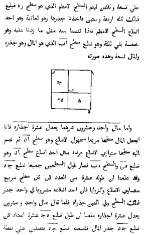

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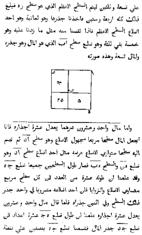

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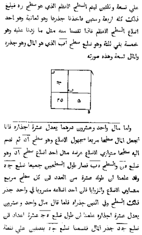

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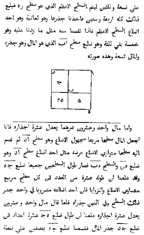

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$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

Al-Khwarizmi's geometric proof of his method

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x^2 + 10x &= 39 \\
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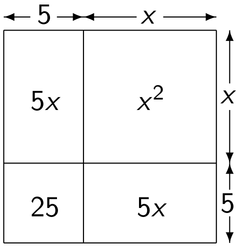

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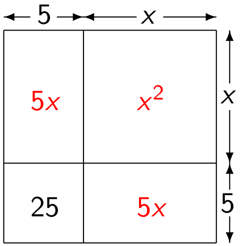

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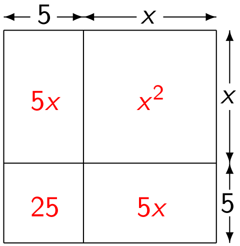

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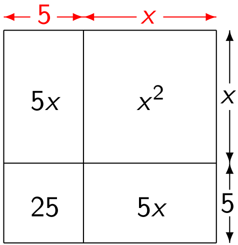

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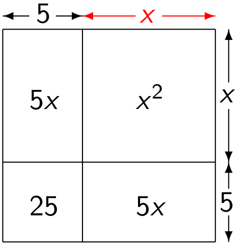

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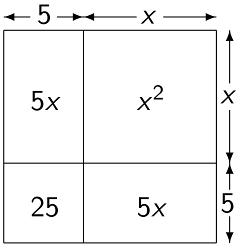

Exercise

Draw a diagram that illustrates the solution of the equation

$$x^2 = 39 + 10x.$$

Answer Skip

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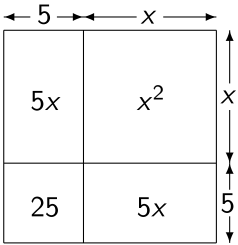

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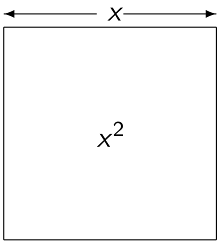

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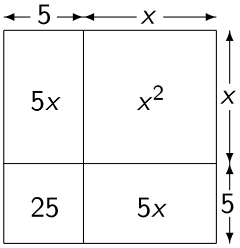

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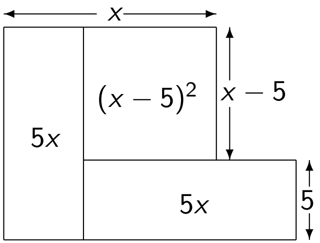

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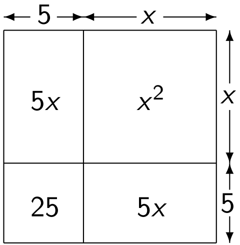

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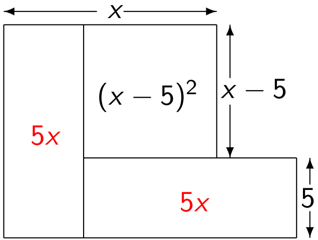

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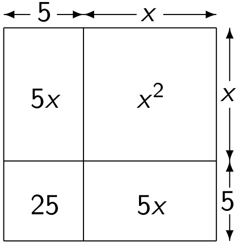

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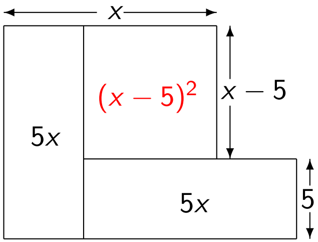

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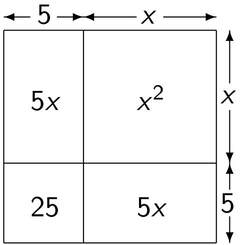
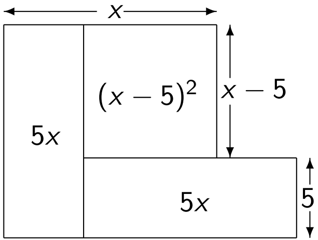


$$\begin{aligned} x^2 &= 39 + 10x \\ x^2 + 25 &= 64 + 10x \\ x - 5 &= 8 \\ x &= 13 \end{aligned}$$



Sample activity from algebra course for middle school teachers

Problem

The expression

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right)$$

is the contribution to a student's final score from three test scores t_1, t_2, and t_3. What is a different way of writing this? Which way should a student use in order to

- calculate the total test contribution to their final grade
- calculate the effect of getting 10 more points on test 2

Responses

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right),\ 0.2t_1+0.2t_2+0.2t_3,\ \frac{t_1}{5}+\frac{t_2}{5}+\frac{t_3}{5},\ \ldots$$

(1) $0.6\left(\frac{t_1+t_2+t_3}{3}\right)$ (2) $0.2t_1+0.2t_2+0.2t_3$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

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Student C: But you can write division as multiplication. Just write it as multiplication by $1/3$.
Student A: Oh yeah! [Discussion shifts to associative law.]

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.

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Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.

$$\begin{aligned} x^2 &= 4 + 9 - 2 \cdot 6 \cos\theta \\ &= 25 + 36 + 2 \cdot 30 \cos\theta \end{aligned}$$

Demo

Ptolemy's theorem

$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd}$$

Ptolemy's theorem

$$\begin{aligned} x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\ &= \frac{bbcd + aacd + abcc + abdd}{ab + cd} \end{aligned}$$

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Theorem (Ptolemy's Theorem)

$$xy = ac + bd$$

Quadratic equation $t^2 + at + b = 0$

$$y + ux + v = 0$$

$$x = t, y = t^2; \qquad u = a, v = b.$$

Quadratic equation $t^2 + at + b = 0$

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Two ways of seeing:

- In xy-space, a parameterized curve and a line.
- In uv-space, a parameterized family of lines and point

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$$y + ux + v = 0$$

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Two ways of seeing:

- In xy-space, a parameterized curve and a line.
- In uv-space, a parameterized family of lines and point

Normal curve

$$\begin{aligned} t^2 + ut + v &= 0 \\ 2t + u &= 0 \\ v &= \frac{u^2}{4} \end{aligned}$$