

Algebra as a Dynamic Environment

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The Five Lessons of This Talk

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Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

... what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.



على تسعة وثلاثين فبمضروب الخمسة مائة وستة وستين
 ذلك كله أربعة وستين فاضافة جذورها وهو ثمانية وستون
 المربع الأصلي المثلثون فإذنا نقصنا منه مثل ما زادنا عليه وهو
 خمسة على ثلثه وهو مبلغ تسعة وأربعين هو الباق وهو جذور
 الباق تسعة وعشرون



وبذا بال واحد وعشرون مبرهما بقدر عشرة أجزاء فإذنا
 جعلنا الباق مضافا مبرها سبعون المربع وهو مبلغ آخر ثم جعلنا
 إليه مضافا مبرها الباق مبرها مائة مثل أحد المربع تسعة وأربعون
 مبلغ من المبلغ مائة مائة مائة المثلثين جميعا مبلغ مائة
 وقد علمنا ان مبرها مبرها من العدد فن كن مبلغ مبرها
 مضروب المبلغ والمبرها إلى أحد المثلثين مبرها إلى واحد مبرها
 فقلت المبلغ إلى الباق مبرها فمما قال مثل واحد مبرها مبرها
 يعدل مبرها مبرها فمما ان مثل مبلغ مبرها مبرها فإذنا
 مبلغ مبرها مبرها فمما مبرها مبرها مبرها مبرها مبرها مبرها

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على تسعة وثلاثين ليم السطح الاثني عشر هو سطح زه فليج
 ذلك انه ليمه ريسين فاحدة جذريا هو ثمانية وهو احد
 السطح الاثني الاثني عشر فانا نفقسه من مثل ما زيدا عليه وهو
 خمسة على ثلثة وهو سطح اربع الذي هو الال وهو جذره
 والال تسعة وهذه مبراهة



ولما بال واحد وعشرون مبراهة فليج عشرة اجزاء فانا
 اجعل الال سطحا مبراهة السطح الاثني عشر وهو سطح اربع ثم نجسم
 اليه سطحا مبراهة السطح اربعة مثل احد السطح اربع اربع وهو
 سطح من السطح اربعة فصار الال السطحين جميعا سطح اربعة
 وقد علمنا ان ثلثة عشرة من العدد من ثلثة سطح مبراهة
 مضروب السطح اربعة والال احد اقله مبراهة الي واحد جذره
 فليج السطح اربعة والال السطح اربعة فليج ثمانية وال واحد وعشرون
 يعدل عشرة اجزاء فليج الال ثلثة عشر اربعة عشر اربعة فليج
 سطح اربعة عشر الال فليج ثمانية عشر اربعة عشر اربعة فليج ثمانية

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

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على تسعة وثلاثين قيم المربع الواحد وهو مائة
 ذلك أنه إذا ضربت في مائة جذوراً فهو مائة وهو مائة
 المربع الواحد المثلث مائة فلهذا من مائة ما إذا كان عليه وهو
 خمسة على ثلثه وهو مائة مائة الذي هو المثل وهو جذور
 والمثل تسعة وهو مائة



وإذا كان واحد وعشرون مائة مائة عشرة أجزاء مائة
 تجعل المثل مائة مائة مائة مائة مائة مائة مائة مائة
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$$x^2 + 10x = 39$$

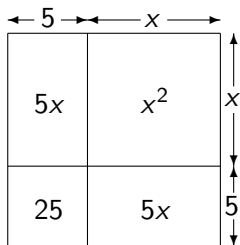
$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

Al-Khwarizmi's geometric proof of his method

$$\begin{aligned}x^2 + 10x &= 39 \\x^2 + 10x + 25 &= 39 + 25 = 64 \\x + 5 &= 8 \\x &= 3\end{aligned}$$



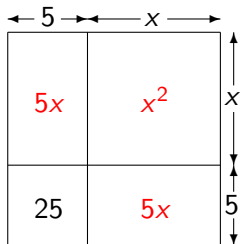
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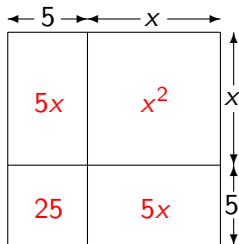
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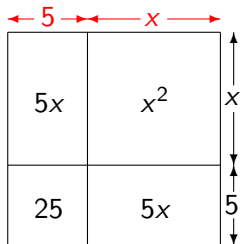
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$$x = 3$$



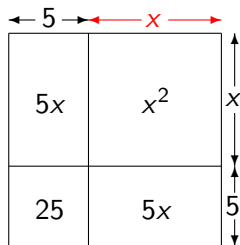
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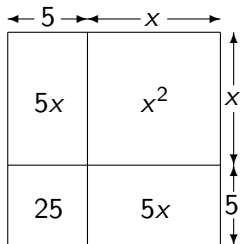
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Exercise

Draw a diagram that illustrates the solution of the equation

$$x^2 = 39 + 10x.$$

▶ Answer

▶ Skip

Al-Khwarizmi's geometric proof of his method

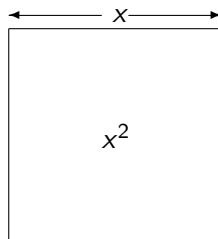
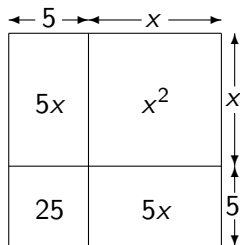
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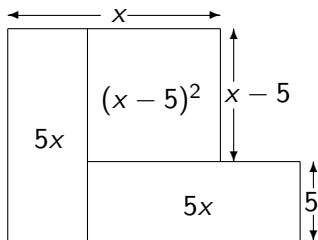
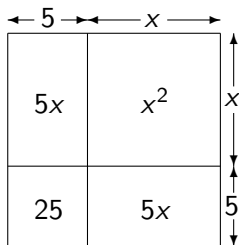
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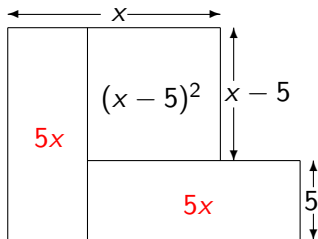
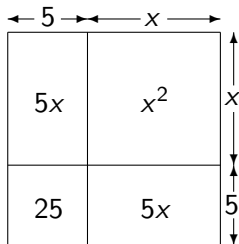
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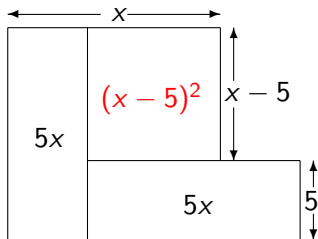
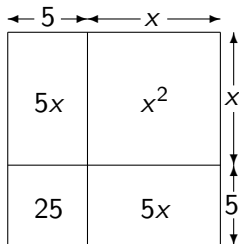
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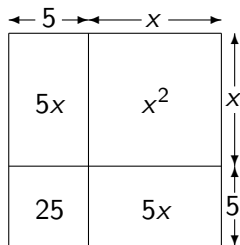
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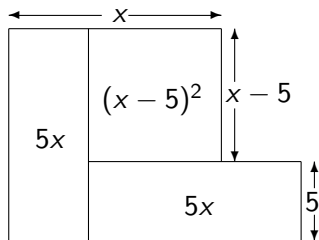


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$$\begin{aligned}x^2 &= 39 + 10x \\x^2 + 25 &= 64 + 10x \\x - 5 &= 8 \\x &= 13\end{aligned}$$



Sample activity from algebra course for middle school teachers

Problem

The expression

$$0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right)$$

is the contribution to a student's final score from three test scores t_1 , t_2 , and t_3 . What is a different way of writing this? Which way should a student use in order to

- ▶ calculate the total test contribution to their final grade
- ▶ calculate the effect of getting 10 more points on test 2

Responses

$$0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right), 0.2t_1 + 0.2t_2 + 0.2t_3, \frac{t_1}{5} + \frac{t_2}{5} + \frac{t_3}{5}, \dots$$

$$(1) 0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right) \quad (2) 0.2t_1 + 0.2t_2 + 0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

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Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.

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Student C: But you can write division as multiplication. Just write it as multiplication by $1/3$.

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Student A: Oh yeah! [Discussion shifts to associative law.]

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.

$$\begin{aligned}x^2 &= 4 + 9 - 2 \cdot 6 \cos \theta \\ &= 25 + 36 + 2 \cdot 30 \cos \theta\end{aligned}$$

Demo

Ptolemy's theorem

$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd}$$

Ptolemy's theorem

$$\begin{aligned}x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\ &= \frac{bbcd + aacd + abcc + abdd}{ab + cd}\end{aligned}$$

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Theorem (Ptolemy's Theorem)

$$xy = ac + bd$$

Quadratic equation $t^2 + at + b = 0$

$$y + ux + v = 0$$

$$x = t, y = t^2; \quad u = a, v = b.$$

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Two ways of seeing:

- ▶ In xy -space, a parameterized curve and a line.
- ▶ In uv -space, a parameterized family of lines and point

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Two ways of seeing:

- ▶ In xy -space, a parameterized curve and a line.
- ▶ In uv -space, a parameterized family of lines and point

Normal curve

$$t^2 + ut + v = 0$$

$$2t + u = 0$$

$$v = \frac{u^2}{4}$$