Algebra as a Dynamic Environment

William McCallum

The University of Arizona

Σ P Δ 2009

The Five Lessons of This Talk

- Algebra is not about symbols, but about the operations of the mind afforded by symbols.

The Five Lessons of This Talk

- Algebra is not about symbols, but about the operations of the mind afforded by symbols.
- Geometry supports algebraic manipulation but also limits it.

The Five Lessons of This Talk

- Algebra is not about symbols, but about the operations of the mind afforded by symbols.
- Geometry supports algebraic manipulation but also limits it.
- Even the simplest manipulations are complex.

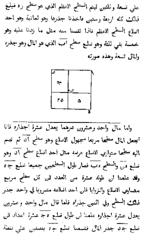
The Five Lessons of This Talk

- Algebra is not about symbols, but about the operations of the mind afforded by symbols.
- Geometry supports algebraic manipulation but also limits it.
- Even the simplest manipulations are complex.
- Even the most complex manipulations can be made simple (except the ones that can't, and those should be done by a computer).

The Five Lessons of This Talk

- Algebra is not about symbols, but about the operations of the mind afforded by symbols.
- Geometry supports algebraic manipulation but also limits it.
- Even the simplest manipulations are complex.
- Even the most complex manipulations can be made simple (except the ones that can't, and those should be done by a computer).
- Algebra is not about symbols, but about the operations of the mind afforded by symbols.

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

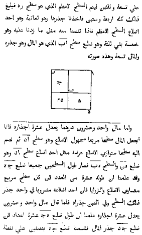


. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830



. . . *what is the square which combined with ten of its roots will give a sum total of 39?* *The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.*

$$x^2 + 10x = 39$$

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

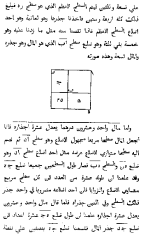
علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع دن والسطح دب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين على نقطة

. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

$$x^2 + 10x = 39$$

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830



. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع دن والسطح دب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين على نقطة

. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع دن والسطح دب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين على نقطة

. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

$$x^2 + 10x = 39$$

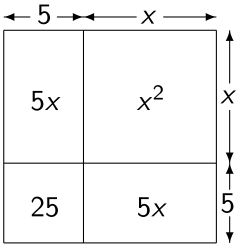
$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

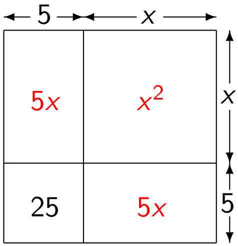
Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$



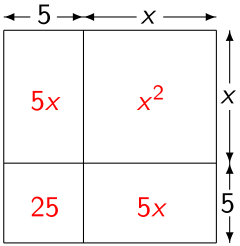
Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$



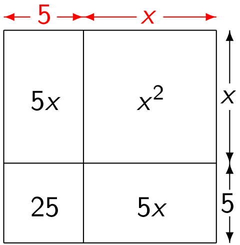
Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$



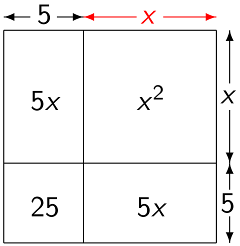
Al-Khwarizmi's geometric proof of his method

$$
\begin{aligned}
x^2 + 10x &= 39 \\
x^2 + 10x + 25 &= 39 + 25 = 64 \\
x + 5 &= 8 \\
x &= 3
\end{aligned}
$$



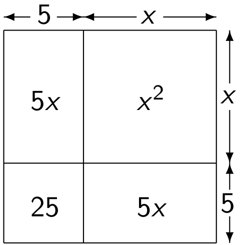
Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$



Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$



Exercise

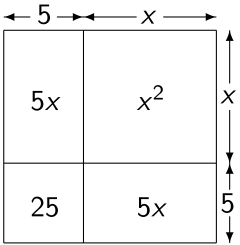
Draw a diagram that illustrates the solution of the equation

$$x^2 = 39 + 10x.$$

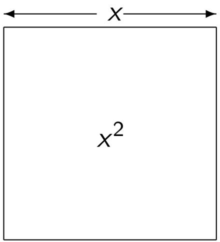
Answer Skip

Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$

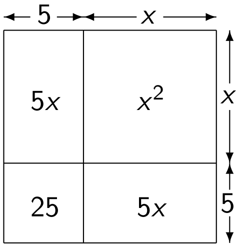


$$x^2 = 39 + 10x$$

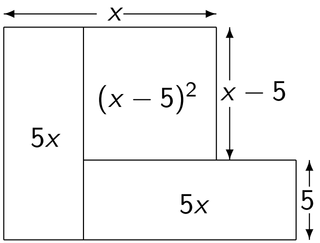


Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$

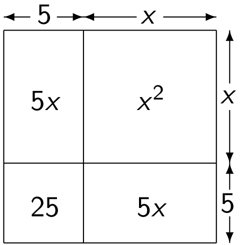


$$\begin{aligned} x^2 &= 39 + 10x \\ x^2 + 25 &= 64 + 10x \end{aligned}$$

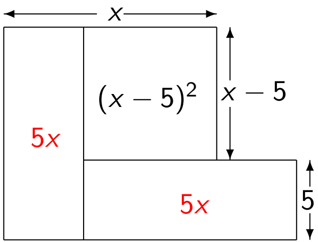


Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$

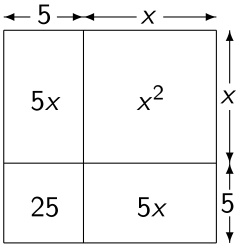


$$\begin{aligned} x^2 &= 39 + 10x \\ x^2 + 25 &= 64 + 10x \end{aligned}$$

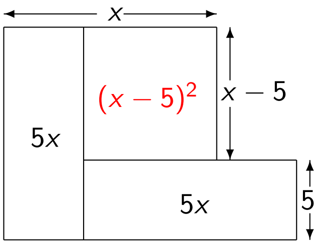


Al-Khwarizmi's geometric proof of his method

$$
\begin{aligned}
x^2 + 10x &= 39 \\
x^2 + 10x + 25 &= 39 + 25 = 64 \\
x + 5 &= 8 \\
x &= 3
\end{aligned}
$$

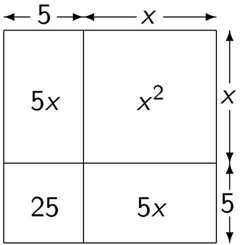


$$
\begin{aligned}
x^2 &= 39 + 10x \\
x^2 + 25 &= {\color{red}64} + 10x
\end{aligned}
$$

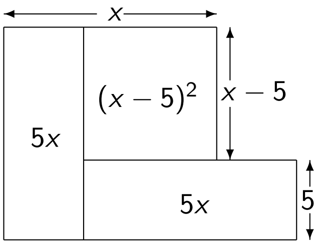


Al-Khwarizmi's geometric proof of his method

$$\begin{aligned} x^2 + 10x &= 39 \\ x^2 + 10x + 25 &= 39 + 25 = 64 \\ x + 5 &= 8 \\ x &= 3 \end{aligned}$$



$$\begin{aligned} x^2 &= 39 + 10x \\ x^2 + 25 &= 64 + 10x \\ x - 5 &= 8 \\ x &= 13 \end{aligned}$$



Sample activity from algebra course for middle school teachers

Problem

The expression

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right)$$

is the contribution to a student's final score from three test scores t_1, t_2, and t_3. What is a different way of writing this? Which way should a student use in order to

- calculate the total test contribution to their final grade
- calculate the effect of getting 10 more points on test 2

Responses

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right),\ 0.2t_1+0.2t_2+0.2t_3,\ \frac{t_1}{5}+\frac{t_2}{5}+\frac{t_3}{5},\ \ldots$$

(1) $0.6\left(\frac{t_1+t_2+t_3}{3}\right)$ (2) $0.2t_1+0.2t_2+0.2t_3$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

$$(1)\ 0.6\left(\frac{t_1+t_2+t_3}{3}\right) \qquad (2)\ 0.2t_1+0.2t_2+0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!

$$(1)\ 0.6\left(\frac{t_1+t_2+t_3}{3}\right) \qquad (2)\ 0.2t_1+0.2t_2+0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!
Student A: How you can see that just by looking at them?

$$(1)\ 0.6\left(\frac{t_1 + t_2 + t_3}{3}\right) \qquad (2)\ 0.2t_1 + 0.2t_2 + 0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!
Student A: How you can see that just by looking at them?
Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.

$$(1)\ 0.6\left(\frac{t_1+t_2+t_3}{3}\right) \qquad (2)\ 0.2t_1+0.2t_2+0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!
Student A: How you can see that just by looking at them?
Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.
Instructor: How do you know you can move the 3 over? What rule says you can do that?

$$(1)\ 0.6\left(\frac{t_1+t_2+t_3}{3}\right) \qquad (2)\ 0.2t_1+0.2t_2+0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

Student B: (1) and (2) are obviously the same!

Student A: How you can see that just by looking at them?

Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.

Instructor: How do you know you can move the 3 over? What rule says you can do that?

Student B: Isn't it because you only have division and multiplication, so it's the commutative law?

$$(1)\ 0.6\left(\frac{t_1+t_2+t_3}{3}\right) \qquad (2)\ 0.2t_1+0.2t_2+0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!
Student A: How you can see that just by looking at them?
Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.
Instructor: How do you know you can move the 3 over? What rule says you can do that?
Student B: Isn't it because you only have division and multiplication, so it's the commutative law?
Instructor: But division isn't commutative.

$$(1)\ 0.6\left(\frac{t_1+t_2+t_3}{3}\right) \qquad (2)\ 0.2t_1+0.2t_2+0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!
Student A: How you can see that just by looking at them?
Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.
Instructor: How do you know you can move the 3 over? What rule says you can do that?
Student B: Isn't it because you only have division and multiplication, so it's the commutative law?
Instructor: But division isn't commutative.
Student C: But you can write division as multiplication. Just write it as multiplication by $1/3$.

$$(1)\ 0.6\left(\frac{t_1+t_2+t_3}{3}\right) \qquad (2)\ 0.2t_1+0.2t_2+0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!
Student A: How you can see that just by looking at them?
Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.
Instructor: How do you know you can move the 3 over? What rule says you can do that?
Student B: Isn't it because you only have division and multiplication, so it's the commutative law?
Instructor: But division isn't commutative.
Student C: But you can write division as multiplication. Just write it as multiplication by $1/3$.
Student A: Oh yeah! [Discussion shifts to associative law.]

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.

$$
\begin{aligned}
x^2 &= 4 + 9 - 2 \cdot 6 \cos\theta \\
&= 25 + 36 + 2 \cdot 30 \cos\theta
\end{aligned}
$$

Demo

Ptolemy's theorem

$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd}$$

Ptolemy's theorem

$$\begin{aligned} x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\ &= \frac{bbcd + aacd + abcc + abdd}{ab + cd} \end{aligned}$$

Ptolemy's theorem

$$\begin{aligned}
x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\
&= \frac{bbcd + aacd + abcc + abdd}{ab + cd} \\
&= \frac{(bc)(bd) + (ac)(ad) + (ac)(bc) + (ad)(bd)}{ab + cd}
\end{aligned}$$

Ptolemy's theorem

$$\begin{aligned}
x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\
&= \frac{bbcd + aacd + abcc + abdd}{ab + cd} \\
&= \frac{(bc)(bd) + (ac)(ad) + (ac)(bc) + (ad)(bd)}{ab + cd} \\
&= \frac{(bd)(bc) + (ac)(ad) + (ac)(bc) + (bd)(ad)}{ab + cd}
\end{aligned}$$

Ptolemy's theorem

$$
\begin{aligned}
x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\
&= \frac{bbcd + aacd + abcc + abdd}{ab + cd} \\
&= \frac{(bc)(bd) + (ac)(ad) + (ac)(bc) + (ad)(bd)}{ab + cd} \\
&= \frac{(bd)(bc) + (ac)(ad) + (ac)(bc) + (bd)(ad)}{ab + cd} \\
&= \frac{(ac)(ad) + (ac)(bc) + (bd)(ad) + (bd)(bc)}{ab + cd}
\end{aligned}
$$

Ptolemy's theorem

$$\begin{aligned}
x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\
&= \frac{bbcd + aacd + abcc + abdd}{ab + cd} \\
&= \frac{(bc)(bd) + (ac)(ad) + (ac)(bc) + (ad)(bd)}{ab + cd} \\
&= \frac{(bd)(bc) + (ac)(ad) + (ac)(bc) + (bd)(ad)}{ab + cd} \\
&= \frac{(ac)(ad) + (ac)(bc) + (bd)(ad) + (bd)(bc)}{ab + cd} \\
&= \frac{(ac + bd)(ad + bc)}{ab + cd}
\end{aligned}$$

Ptolemy's theorem

$$
\begin{aligned}
x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\
&= \frac{bbcd + aacd + abcc + abdd}{ab + cd} \\
&= \frac{(bc)(bd) + (ac)(ad) + (ac)(bc) + (ad)(bd)}{ab + cd} \\
&= \frac{(bd)(bc) + (ac)(ad) + (ac)(bc) + (bd)(ad)}{ab + cd} \\
&= \frac{(ac)(ad) + (ac)(bc) + (bd)(ad) + (bd)(bc)}{ab + cd} \\
&= \frac{(ac + bd)(ad + bc)}{ab + cd}
\end{aligned}
$$

Theorem (Ptolemy's Theorem)

$$xy = ac + bd$$

Quadratic equation $t^2 + at + b = 0$

$$y + ux + v = 0$$

$$x = t, y = t^2; \qquad u = a, v = b.$$

Quadratic equation $t^2 + at + b = 0$

$$y + ux + v = 0$$

$$x = t, y = t^2; \qquad u = a, v = b.$$

Two ways of seeing:

- In xy-space, a parameterized curve and a line.
- In uv-space, a parameterized family of lines and point

Quadratic equation $t^2 + at + b = 0$

$$y + ux + v = 0$$

$$x = t, y = t^2; \qquad u = a, v = b.$$

Two ways of seeing:

- In xy-space, a parameterized curve and a line.
- In uv-space, a parameterized family of lines and point

Normal curve

$$\begin{aligned} t^2 + ut + v &= 0 \\ 2t + u &= 0 \\ v &= \frac{u^2}{4} \end{aligned}$$