

Algebra as a Dynamic Environment

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The Five Lessons of This Talk

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Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

... what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.



على تسعة وثلاثين فبم المربع الواحد وهو خمسة وعشرون
 ذلك انه اربعة وعشرون ماضة جذريا وهو ثمانية وهو اربعة
 المربع الواحد الماضة ثمانية فبم تسعة ماضة ما زادنا عليه وهو
 خمسة على ثلثة وهو سبع المربع هو الابل وهو جذره
 والابل تسعة وهذه مبراهة



وبدا بال واحد وعشرون مبراهة بقل مبراهة اربعة ماضة
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على تسعة وثلاثين فبمربع الخمسة مائة وستة وستين
 ذلك كله أربعة وخمسة مائة وستة وستين وهو ثمانية وستون
 المربع الأصلي المثلث مائة وستة وستين مائة وستة وستين
 خمسة مائة وستة وستين وهو مربع الخمسة مائة وستة وستين
 والمثل تسعة مائة وستة وستين



وبدا بال واحد وعشرون مائة وستة وستين عشرة مائة وستة وستين
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على تسعة وثلاثين نيم السطح الاثنى عشر اذ هو سطح ذو مربع
 ذلك انه اربعة درجسي فاحدة جذريا وهو ثمانية وهو احد
 السطح الاثنى عشر فاذنا فليسا منه مثل ما زادنا عليه وهو
 خمسة على ثلثة وهو سطح اربع الذي هو الابل وهو جذره
 والابل تسعة وهذه مبرهنة



ولذا بال واحد وعشرون مبرهنة بتدليل عشرة اجزاء فان
 اجعل الابل سطحا مبرهنا السطح الاثنا عشر وهو سطح اربعة
 ايه سطحا سداسي السطح مبرهنة مثل احد السطح اربعة
 سطح من السطح اربعة فسطح الابل السطح اربعة
 وقد علمنا ان سطح مبرهنة من العدد ان كان سطح مربع
 مضروب السطح اربعة والابل احد اقله مبرهنة الابل واحد جذره
 فثلث السطح والابل السطح فلهذا فالابل واحد وعشرون
 يعدل عشرة اجزاء من الابل سطح اربعة عشرة اعداد ان
 سطح اربعة عشرة اعداد فلهذا فلهذا فلهذا فلهذا فلهذا

$$x^2 + 10x = 39$$

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

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على تسعة وثلاثين نيم السطح الاثنى عشر السطح رة فليج ذلك انه لرمه ريسين فاحدة جذورا وهو ثمانية وهو احد السطح الاثنى الاثني عشر فانا نفقسه من مثل ما زادا عليه وهو خمسة على ثلثة وهو سبع سطح ارب الذي هو الابل وهو جذره والابل تسعة وهذه مبرهه



ولما بال واحد وعشرون ضربا بثلثة عشرة اجزاء فانا نجعل الابل مضاعفا مره اربعين لاضاع وهو سطح ارب ثم نجسم اليه سطحا سداسي السطح مره مثل احد السطح ارب وهو سطح من السطح رة فصار طول السطحين جميعا تسعة مائة وقد علمنا ان طول عشرة من العدد فن كان سطح مربع مضاعف السطح واكثرنا اليه احد اقله مديريه الي واحد جدر فثقت السطح وبني السطح فثقتا فاما ثلث مثل واحد وسبعين يعدل عشرة اجزاء فثقتا ان طول سطح ارب عشرة اجزاء فن سطح ارب فثقتا سطح ارب فحصل على ثلثة

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على تسعة وقتين قيم السطح الاكبر الذي هو سطح زه فبقي ذلك كله لزمنه يساوي مساحة جذره هو ثمانية وهو احد السطح الاكبر الاكبر فباقي التسعة مقل ما زاد عليه وهو خمسة في ثلثه وهو سبع السطح الارب الذي هو الاكبر وهو جذره واقل تسعة وهذه هي



وبدا بال واحد وعشرون مرفعا بقدر عشرة اعدادنا تا حصل الاكبر منها السطح الاكبر وهو سطح اربعة وتسعين اية عليها سداسي السطح مره مثل احد السطح اربعة وتسعين ومن السطح ربعه مثل طول السطحين جميعا فبقي سبعه وقد قلنا ان طول عشرة من العدد من ثلثه سطح مربع مضروب السطح واكثرنا الي احد اعدادنا مرفوعا الي واحد جذر فقلت السطح والي السطح فبقا ثلث مثل واحد وعشرين بعدل عشرة اعدادنا من طول سبعه وتسعة اعدادنا من سطح ستة جذر الاكبر تسعة سطح اربعة وسبعين على ثلثه

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

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على تسعة وثلاثين فليس المربع الاثني عشر الواحد هو مربعه
 ذلك انه اربعة وخمسة ماضة جذريا هو ثمانية وهو احد
 المربع الاثنى عشر الاثني ثمانية فلهذا منه مثل ما زاد عليه وهو
 خمسة على ثلثة وهو مربع ثلثة الذي هو الابل وهو جذره
 والابل تسعة وهذه حروقه



ولما كان واحد وعشرون مبرهنا بقابل عشرة اجزاء لنا
 اجعل الابل مضافا مبرهنا الاثني عشر الواحد وهو مبالغ آت لم نجس
 اليه مضافا مبرهنا الاثني عشر واحد مثل احد المربع الثلث آت وهو
 مبلغ من والمبلغ ثمة مضافا لابل المبرهنا جميعا المبلغ ثمة
 وقد قلنا ان ثلثة عشرة من العدد من ثلثة مبالغ مربع
 مضاهيا المبلغ والبرهنا الى احد اقله مبرهنا الى واحد جذره
 فقلت الثلث على اثنى عشر فلهذا ثلثة مبالغ واحد وعشرون
 يعدل عشرة اجزاء منها ان ثلثة مبالغ ثمة عشرة اعداد في
 مبلغ ثمة جذره الابل تسعة مبالغ ثمة مصلين على ثلثة

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على تسعة وثلثين فليم اسلخ الاربعة الاربعة واربعة
 ذلكت انه اربعة وثلثين فاسلخ جذورا فهو ثمانية واربعة
 اسلخ اسلخ الاربعة فانا فقسنا منه مثل ما زادنا عليه وهو
 خمسة على ثلثه وهو سبعة اسلخ ثلث الذي هو اقل وهو جذرة
 وابل تسعة وهذه مبراهة



وبدا بال واحد وعشرون فربعا بقابل عشرة اجزاء فانا
 اجعل الال متفقا مبراهة اسلخ وهو اسلخ آء ثم لحسم
 اليه سلخا سداسي اسلخ مبراهة مثل احد اسلخ آء وهو
 سبعة من اسلخ اء فاسلخ اء فاسلخ اء فاسلخ اء فاسلخ اء
 وقد قلنا ان في هذه عشرة من العدد فن ثلث اسلخ مربع
 مضارب اسلخ واربعة الى احد اقله مبراهة في واحد جدر
 فثقت اسلخ ولى الالتي جذره فلما قال مثل واحد وعشرون
 يعدل عشرة اجزاء فملأ ان مثل سبعة واربعة عشرة اعداد فن
 سلخ جاذر لعل فثقتنا سلخ آء فحصل على ثمانية

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$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

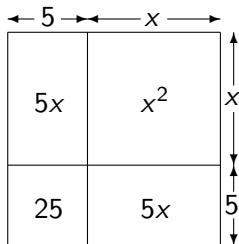
Al-Khwarizmi's geometric proof of his method

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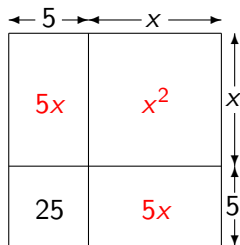
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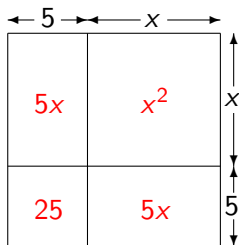
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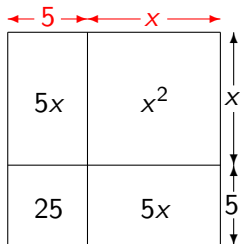
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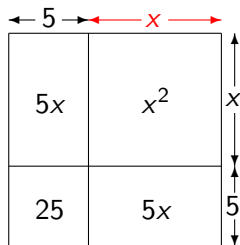
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$$\begin{aligned}x^2 + 10x &= 39 \\x^2 + 10x + 25 &= 39 + 25 = 64 \\x + 5 &= 8 \\x &= 3\end{aligned}$$



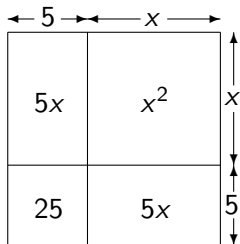
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Exercise

Draw a diagram that illustrates the solution of the equation

$$x^2 = 39 + 10x.$$

▶ Answer

▶ Skip

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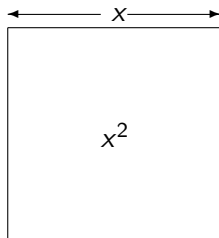
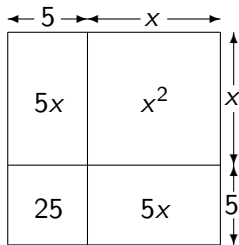
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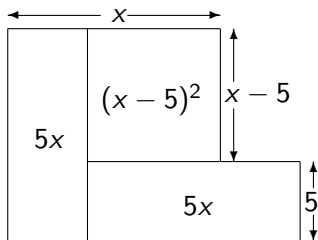
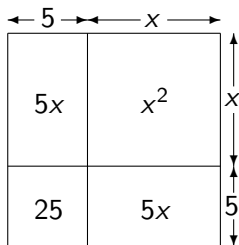
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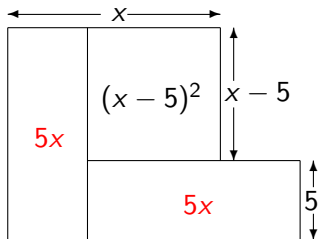
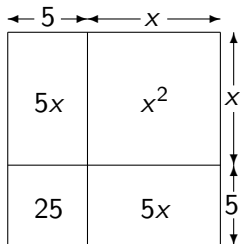
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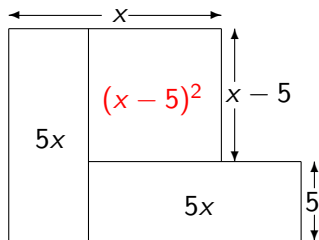
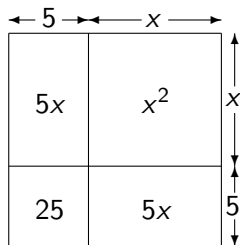
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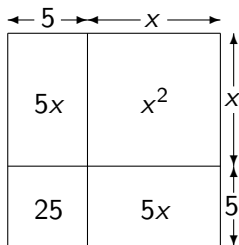
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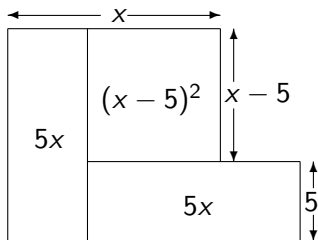


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$$\begin{aligned}x^2 + 10x &= 39 \\x^2 + 10x + 25 &= 39 + 25 = 64 \\x + 5 &= 8 \\x &= 3\end{aligned}$$



$$\begin{aligned}x^2 &= 39 + 10x \\x^2 + 25 &= 64 + 10x \\x - 5 &= 8 \\x &= 13\end{aligned}$$



Sample activity from algebra course for middle school teachers

Problem

The expression

$$0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right)$$

is the contribution to a student's final score from three test scores t_1 , t_2 , and t_3 . What is a different way of writing this? Which way should a student use in order to

- ▶ calculate the total test contribution to their final grade
- ▶ calculate the effect of getting 10 more points on test 2

Responses

$$0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right), 0.2t_1 + 0.2t_2 + 0.2t_3, \frac{t_1}{5} + \frac{t_2}{5} + \frac{t_3}{5}, \dots$$

$$(1) 0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right) \quad (2) 0.2t_1 + 0.2t_2 + 0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

$$(1) 0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right) \quad (2) 0.2t_1 + 0.2t_2 + 0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

Student B: (1) and (2) are obviously the same!

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Student A: How you can see that just by looking at them?

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Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

Student B: (1) and (2) are obviously the same!

Student A: How can you see that just by looking at them?

Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.

$$(1) 0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right) \quad (2) 0.2t_1 + 0.2t_2 + 0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

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Student A: How can you see that just by looking at them?

Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.

Instructor: How do you know you can move the 3 over? What rule says you can do that?

$$(1) 0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right) \quad (2) 0.2t_1 + 0.2t_2 + 0.2t_3$$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

Student B: (1) and (2) are obviously the same!

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Student A: Oh yeah! [Discussion shifts to associative law.]

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.

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$$\begin{aligned}x^2 &= 4 + 9 - 2 \cdot 6 \cos \theta \\ &= 25 + 36 + 2 \cdot 30 \cos \theta\end{aligned}$$

Demo

Ptolemy's theorem

$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd}$$

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Theorem (Ptolemy's Theorem)

$$xy = ac + bd$$

Quadratic equation $t^2 + at + b = 0$

$$y + ux + v = 0$$

$$x = t, y = t^2; \quad u = a, v = b.$$

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- ▶ In xy -space, a parameterized curve and a line.
- ▶ In uv -space, a parameterized family of lines and point

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Normal curve

$$t^2 + ut + v = 0$$

$$2t + u = 0$$

$$v = \frac{u^2}{4}$$