Where’s the Algebra?

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Conversation Among Colleagues
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...what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.
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\[ x + 5 = 8 \]
\[ x = 3 \]
The quadratic formula in the 17th century

From the Oxford Museum of History of Science (Stephen Johnston, photo Bluebridge Farm Studio)
the students are faced with a bewildering variety of processes which they repeat by rote in order to master them. The learning is almost always sheer memorization. 

as far as the students can see the topics are unrelated. They are like pages torn from a hundred different books, no one of which conveys the life, meaning and spirit of mathematics. This presentation of algebra begins nowhere and ends nowhere.
\[
\frac{x + 2}{x} = 2
\]

\[
\frac{x}{2} - \frac{3}{2} = x - 3
\]
Instructional programs from prekindergarten through grade 12 should enable all students to-

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships;
- analyze change in various contexts.
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Different machines can produce the same function:

\[ x \rightarrow 4x^2 \quad \text{and} \quad p \rightarrow (2p)^2 \]
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Algebra can describe the machinery inside the black box
What is algebra about?

- Functions are a useful vehicle for teaching algebra, but algebra is not *about* functions.
- Symbolic manipulation is an important part of the machinery, but algebra is not *about* symbolic manipulation.
- Algebra is about reasoning with numbers and operations. This includes working with
  - symbols
  - expressions
  - equations
  - functions.
A group of fourth graders is discussing a subtraction problem:

\[145 - 98 = 47.\]

They have understood \(145 - 100 = 45\); some understand that to go from there they have have to add 2 to the 45, some think they should subtract it, others would prefer to ignore it. Max, listening to all this, says excitedly:

Yeah, the less you subtract, the more you end up with. AND ... in fact the thing you end up with is exactly as much larger as the amount less that you subtracted.
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$$a - (b - c) = (a - b) + c$$
Sample activity from algebra course

Problem

The expression

$$0.6 \left( \frac{t_1 + t_2 + t_3}{3} \right)$$

is the contribution to a student’s final score from three test scores. What is a different way of writing this? Which way should a student use in order to

- calculate the total test contribution to their final grade
- calculate the effect of getting 10 more points on test 2

Responses

$$0.6 \left( \frac{t_1 + t_2 + t_3}{3} \right), \ 0.2t_1 + 0.2t_2 + 0.2t_3, \ \frac{t_1}{5} + \frac{t_2}{5} + \frac{t_3}{5}, \ldots$$
Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I’m not sure it is right.
(1) \[0.6 \left( \frac{t_1 + t_2 + t_3}{3} \right)\]  
(2) \[0.2t_1 + 0.2t_2 + 0.2t_3\]

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I’m not sure it is right.

Student B: (1) and (2) are obviously the same!
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Student A: How you can see that just by looking at them?  
Student B: You just move the 3 over so it’s dividing the 0.6, which gives you 0.2, then distributed the 0.2.
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Instructor: How do you know you can move the 3 over? What rule says you can do that?
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Instructor: But division isn’t commutative.
Student C: But you can write division as multiplication. Just write it as multiplication by 1/3.
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Student C: But you can write division as multiplication. Just write it as multiplication by \(\frac{1}{3}\).
Student A: Oh yeah! [Discussion shifts to associative law.]
Viete’s formulas and the quadratic formula

If

\[ ax^2 + bx + c = 0 \]

then let

\[ r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]
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Exercise

Give an explanation, purely in terms of the structure of the expressions, of why these two numbers satisfy

\[ r + s = -\frac{b}{a} \quad \text{and} \quad rs = \frac{c}{a}. \]
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**Answer**

When you add \( r \) and \( s \), the plus and minus signs cancel.
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If

$$ax^2 + bx + c = 0$$

then let

$$r = \frac{-b}{2a} \quad \text{and} \quad s = \frac{-b}{2a}$$

Answer

When you add $r$ and $s$, the plus and minus signs cancel.
Viete’s formulas and the quadratic formula

If

\[ ax^2 + bx + c = 0 \]

then let

\[ r = \frac{2a}{2a} \quad \text{and} \quad s = \frac{-2a}{2a} \]

**Answer**

When you add \( r \) and \( s \), the plus and minus signs cancel. When you multiply \( r \) and \( s \), you get the difference of two squares in the numerator,

\[ (-b)^2 - (\sqrt{b^2 - ac})^2 = b^2 - (b^2 - 4c) = 4c. \]
Given a cyclic quadrilateral whose sides are 2, 3, 5, 6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3 (Askey).
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\[ x^2 = 4 + 9 - 2 \cdot 6 \cos \theta \]
\[ = 25 + 36 + 2 \cdot 30 \cos \theta \]
Ptolemy’s theorem

\[ x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \]
Ptolemy’s theorem

\[ x^2 = \frac{b^2 cd + a^2 cd + abc^2 + abd^2}{ab + cd} \]

\[ = \frac{(ac + bd)(ad + bc)}{ab + cd} \]
Ptolemy’s theorem

\[ x^2 = \frac{b^2 cd + a^2 cd + abc^2 + abd^2}{ab + cd} = \frac{(ac + bd)(ad + bc)}{ab + cd} \]

**Theorem (Ptolemy’s Theorem)**

\[ xy = ac + bd \]
Expressions

1. (a) Write an algebraic expression representing each of the following operations on a number \( b \):

   “Multiply by 0.4”
   “Divide by five-halves”

   (b) Are these expressions equivalent? What does this tell you?

2. To convert from miles to kilometers, Abby takes the number of miles, doubles it, then subtracts 20% from the result. Renato first divides the number of miles by 5, and then multiplies the result by 8.

   (a) Write an algebraic expression for each method.
   (b) Use your answer to part (a) to decide if the two methods give the same answer.
In the following problems, the solution to the equation depends on the constant $a$. Assuming $a$ is positive, what is the effect of increasing $a$ on the value of the solution? Does the solution increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

1. $x - a = 0$.

2. $ax = 1$.

3. $ax = a$.

4. $\frac{x}{a} = 1$. 
In the following problems, the solution to the equation depends on the constant $a$. Assuming $a$ is positive, what is the effect of increasing $a$ on the value of the solution? Does the solution increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

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   *Increases.* The larger $a$ is, the larger $x$ must be to give 0.

2. $ax = 1$.

3. $ax = a$.

4. $\frac{x}{a} = 1$. 
   (The answer here is a bit unusual as $a$ is not equal to 0, so the statement itself is not a complete explanation, but this is how the question is presented.)
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1. $x - a = 0$. *Increases*. *The larger $a$ is, the larger $x$ must be to give 0.*

2. $ax = 1$. *Decreases*. *The larger $a$ is, the smaller $x$ must be to give a product of 1.*

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3. $ax = a$. Remains unchanged. As $a$ changes, the two sides of the equation change together and remain equal.
4. $\frac{x}{a} = 1$. Increases. The larger $a$ is, the larger $x$ must be to give a ratio of 1.