Where's the Algebra?

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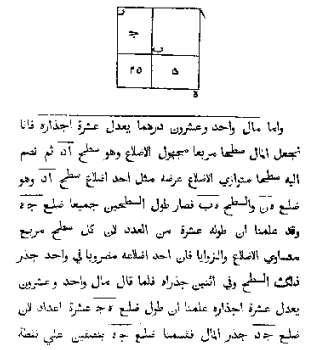
University of Arizona

Conversation Among Colleagues
Western Michigan University
March 2008

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830



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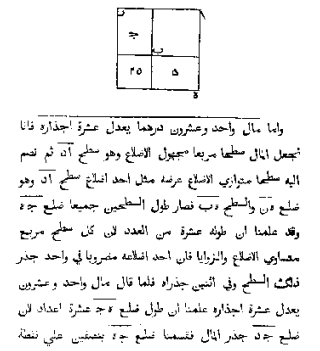

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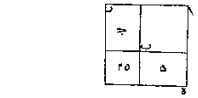

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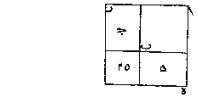

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ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
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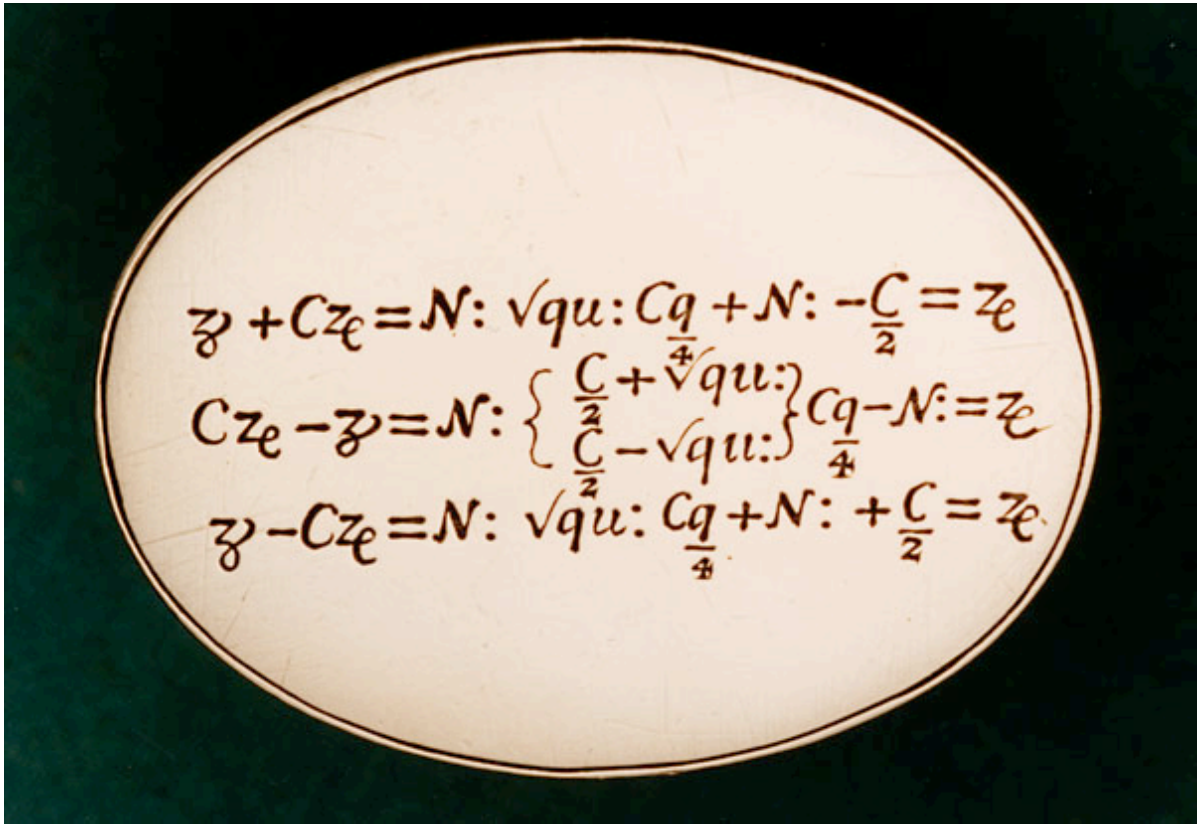
$$x + 5 = 8$$

$$x = 3$$

The quadratic formula in the 17th century



From the Oxford Museum of History of Science (Stephen Johnston, photo Bluebridge Farm Studio)

Why Johnny Can't Add (1973)

. . . the students are faced with a bewildering variety of processes which they repeat by rote in order to master them. The learning is almost always sheer memorization.

. . . as far as the students can see the topics are unrelated. They are like pages torn from a hundred different books, no one of which conveys the life, meaning and spirit of mathematics. This presentation of algebra begins nowhere and ends nowhere.

$$\frac{x+2}{x} = 2$$

$$\frac{x}{2} - \frac{3}{2} = x - 3$$

Principles and Standards of School Mathematics (2000)

Instructional programs from prekindergarten through grade 12 should enable all students to-

- *understand patterns, relations, and functions;*
- *represent and analyze mathematical situations and structures using algebraic symbols;*
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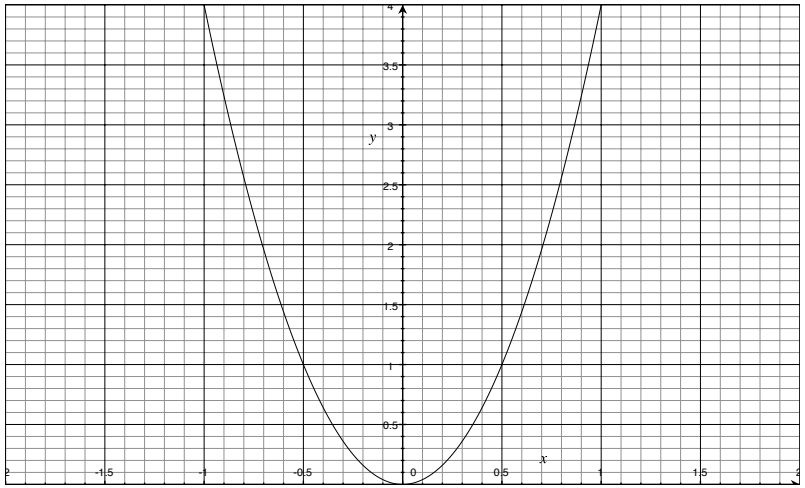
$$x \longrightarrow 4x^2 \quad \text{and} \quad p \longrightarrow (2p)^2$$



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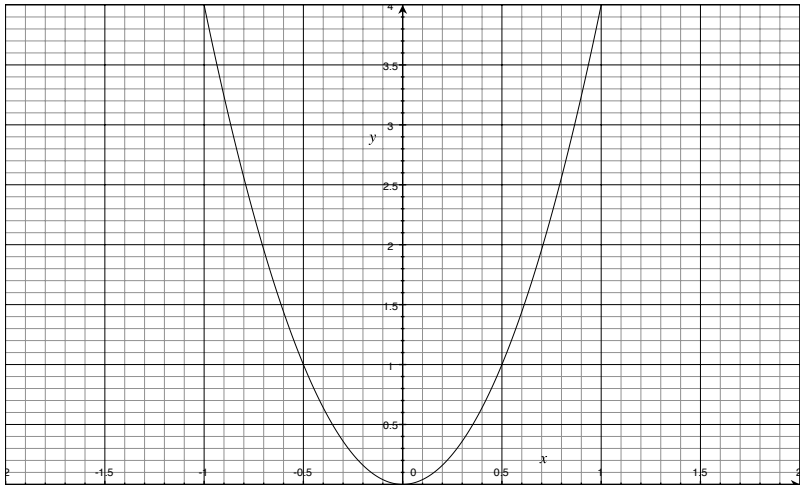

- Algebra can describe the machinery inside the black box

What is algebra about?

- Functions are a useful vehicle for teaching algebra, but algebra is not *about* functions.
- Symbolic manipulation is an important part of the machinery, but algebra is not *about* symbolic manipulation.
- Algebra is about reasoning with numbers and operations. This includes working with
  - symbols
  - expressions
  - equations
  - functions.

Max finds a theorem

A group of fourth graders is discussing a subtraction problem:

$$145 - 98 = 47.$$

They have understood $145 - 100 = 45$; some understand that to go from there they have have to add 2 to the 45, some think they should subtract it, others would prefer to ignore it. Max, listening to all this, says excitedly:

> *Yeah, the less you subtract, the more you end up with. AND . . . in fact the thing you end up with is exactly as much larger as the amount less that you subtracted.*

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$$a - (b - c) = (a - b) + c$$

Sample activity from algebra course

Problem

The expression

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right)$$

is the contribution to a student's final score from three test scores. What is a different way of writing this? Which way should a student use in order to

- calculate the total test contribution to their final grade
- calculate the effect of getting 10 more points on test 2

Responses

$$0.6\left(\frac{t_1+t_2+t_3}{3}\right),\ 0.2t_1+0.2t_2+0.2t_3,\ \frac{t_1}{5}+\frac{t_2}{5}+\frac{t_3}{5},\ \ldots$$

(1) $0.6\left(\frac{t_1+t_2+t_3}{3}\right)$ (2) $0.2t_1+0.2t_2+0.2t_3$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

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Student A: Oh yeah! [Discussion shifts to associative law.]

Viete's formulas and the quadratic formula

If

$$ax^2 + bx + c = 0$$

then let

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

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Exercise

Give an explanation, purely in terms of the structure of the expressions, of why these two numbers satisfy

$$r + s = -\frac{b}{a} \quad \text{and} \quad rs = \frac{c}{a}.$$

Answer

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Answer

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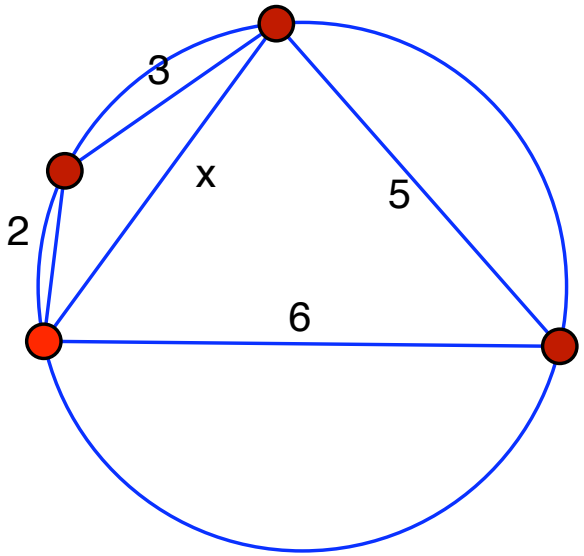
Answer

When you add r and s, the plus and minus signs cancel. When you multiply r and s, you get the difference of two squares in the numerator,

$$(-b)^2 - (\sqrt{b^2 - ac})^2 = b^2 - (b^2 - 4c) = 4c.$$

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3 (Askey).



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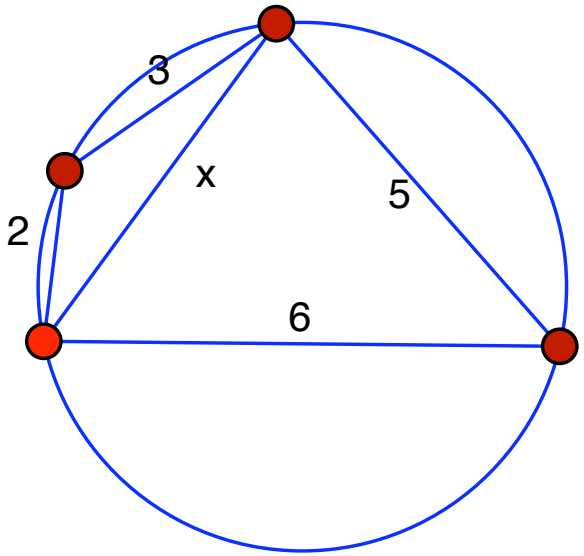
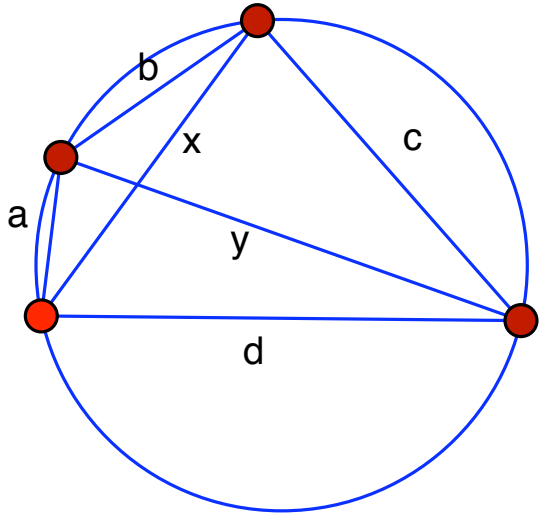


$$\begin{aligned} x^2 &= 4 + 9 - 2 \cdot 6 \cos\theta \\ &= 25 + 36 + 2 \cdot 30 \cos\theta \end{aligned}$$

Demo

Ptolemy's theorem



$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd}$$

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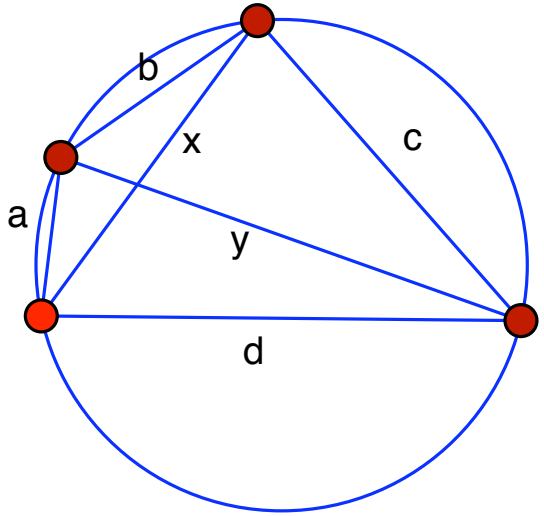

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\begin{aligned}
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&= \frac{(ac + bd)(ad + bc)}{ab + cd}
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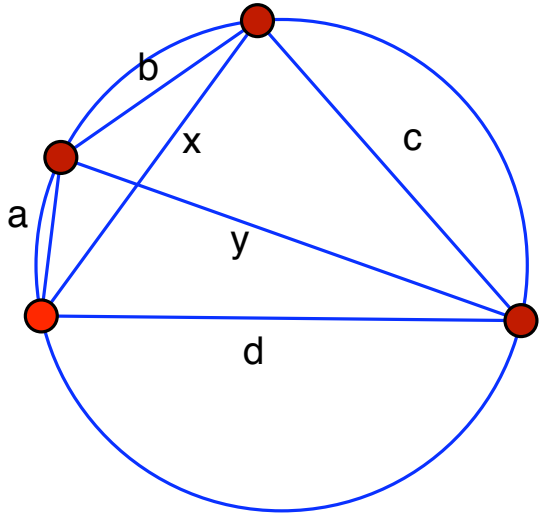

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Theorem (Ptolemy's Theorem)

$$xy = ac + bd$$

Expressions

1.
   (a) Write an algebraic expression representing each of the following operations on a number b:

   "Multiply by 0.4"

   "Divide by five-halves"

   (b) Are these expressions equivalent? What does this tell you?

2. To convert from miles to kilometers, Abby takes the number of miles, doubles it, then subtracts 20% from the result. Renato first divides the number of miles by 5, and then multiplies the result by 8.
   (a) Write an algebraic expression for each method.
   (b) Use your answer to part (a) to decide if the two methods give the same answer.

Equations

In the following problems, the solution to the equation depends on the constant a. Assuming a is positive, what is the effect of increasing a on the value of the solution? Does the solution increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

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2. $ax = 1.$
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4. $\dfrac{x}{a} = 1.$

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2. $ax = 1$. *Decreases. The larger a is, the smaller x must be to give a product of 1.*
3. $ax = a$.
4. $\dfrac{x}{a} = 1$.

Equations

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4. $\frac{x}{a} = 1$. *Increases. The larger a is, the larger x must be to give a ratio of 1.*