Symbols and Science

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Cardano, Sponge Cake, and Notation

Cardano's famous book "Ars Magna" (1545) marked the end of the medieval period in mathematics and triggered the development of modern mathematical notation. The famous formula for the cubic, as presented in the "Ars Magna", resembles more a culinary recipe than a modern formula. The switchover has implications for the teaching of mathematics.

One view of algebra: quadratic equations

The quadratic formula

A number x satisfies

$$x^2 + bx + c = 0$$

if and only if

$$x = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Factoring

If $r + s = -b$ and $rs = c$, then

$$x^2 + bx + c = (x - r)(x - s)$$

and

$$x^2 + bx + c = 0 \qquad \text{if and only if} \qquad x = r \quad \text{or} \quad x = s.$$

Another view of algebra: understanding functions

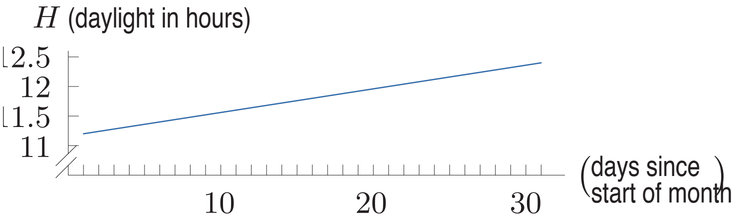

Figure: Sunlight in Madrid

The graph shows the number of hours of daylight in Madrid for one month.

- Why does the graph look linear?
- Estimate and interpret the slope of the line.
- What month does the graph show?

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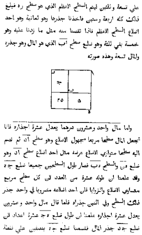
What symbolic skills do modern students need?

- Manipulative skills?
- Graphical and numerical understanding?
- Symbol sense?

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830



. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

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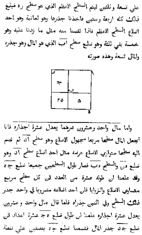
Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830



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Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

على تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو
ضلع دن والسطح دب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين على نقطة

. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

$$x^2 + 10x = 39$$

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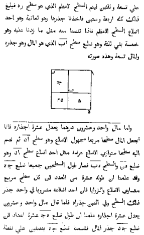

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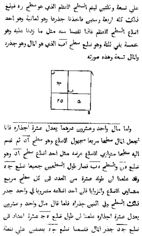

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$$x^2 + 10x + 25 = 39 + 25 = 64$$

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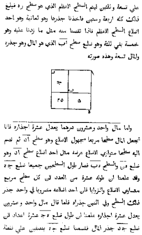

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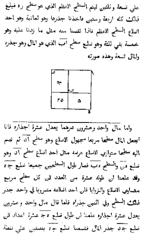

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$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

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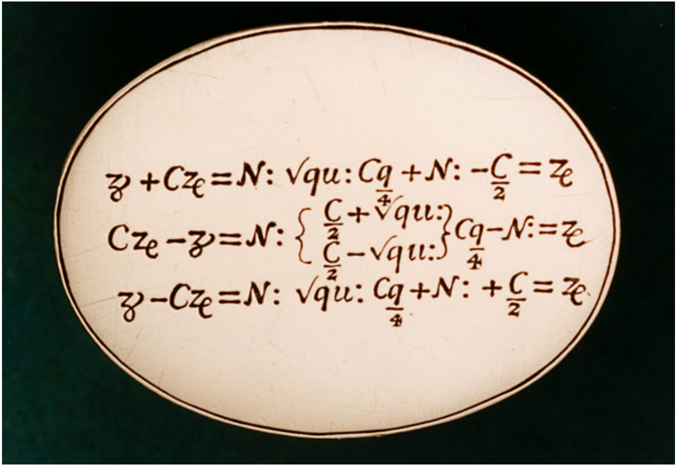
The quadratic formula in the 17th century



From the Oxford Museum of History of Science (Stephen Johnston, photo Bluebridge Farm Studio)

What is going on here?

$$z + Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : -\frac{C}{2} = r$$

$$Cr - z = N : \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \frac{Cq}{4} - N : = r$$

$$z - Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : +\frac{C}{2} = r$$

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What is going on here?

$$x^2 + Cx = N : \sqrt{qu} : \frac{Cq}{4} + N : \ - \frac{C}{2} = x$$

$$Cx - x^2 = N : \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \frac{Cq}{4} - N : \ = x$$

$$x^2 - Cx = N : \sqrt{qu} : \frac{Cq}{4} + N : \ + \frac{C}{2} = x$$

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What is going on here?

$$
\begin{array}{ll}
x^2 + Cx = N, & \sqrt{qu} : \dfrac{C^2}{4} + N : -\dfrac{C}{2} = x \\[2ex]
Cx - x^2 = N, & \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \dfrac{C^2}{4} - N : = x \\[2ex]
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\end{array}
$$

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What is going on here?

$$x^2 + Cx = N, \qquad \sqrt{qu} : \frac{C^2}{4} + N : \; - \frac{C}{2} = x$$

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$$x^2 - Cx = N, \qquad \sqrt{qu} : \frac{C^2}{4} + N : \; + \frac{C}{2} = x$$

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What is going on here?

$$
\begin{aligned}
x^2 + Cx = N, &\qquad \sqrt{\frac{C^2}{4} + N} - \frac{C}{2} = x \\
Cx - x^2 = N, &\qquad \frac{C}{2} \pm \sqrt{\frac{C^2}{4} - N} = x \\
x^2 - Cx = N, &\qquad \sqrt{\frac{C^2}{4} + N} + \frac{C}{2} = x
\end{aligned}
$$

Viete's formulae and the quadratic formula

If

$$x^2 + bx + c = 0$$

then let

$$r = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

Exercise

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Viete's formulae and the quadratic formula

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Exercise

Give an explanation, purely in terms of the structure of the expressions, of why these two numbers satisfy

$$r + s = -b \quad \text{and} \quad rs = c.$$

Answer Skip

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Viete's formulae and the quadratic formula

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Answer

When you add r and s, the plus and minus signs cancel.

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Viete's formulae and the quadratic formula

If

$$x^2 + bx + c = 0$$

then let

$$r = \frac{\boxed{} + \boxed{\phantom{\sqrt{b^2-4c}}}}{2} \quad \text{and} \quad s = \frac{\boxed{} - \boxed{\phantom{\sqrt{b^2-4c}}}}{2}$$

Answer

When you add r and s, the plus and minus signs cancel.

When you multiply r and s, you get the difference of two squares in the numerator,

$$(-b)^2 - (\sqrt{b^2 - 4c})^2 = b^2 - (b^2 - 4c) = 4c.$$

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An example from economics

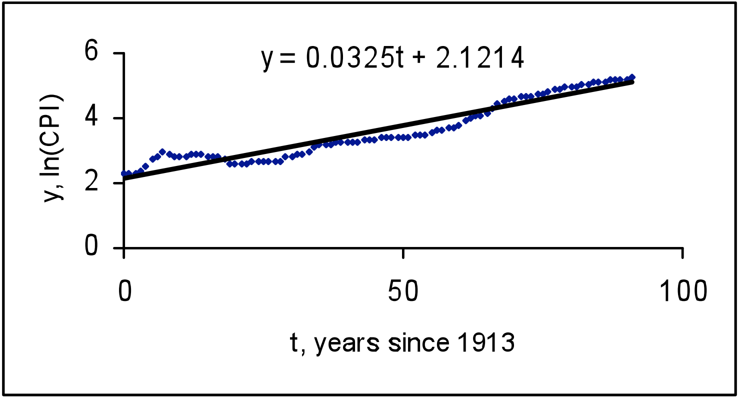

Figure: Cost price index for the last 100 years

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Another view of the same data

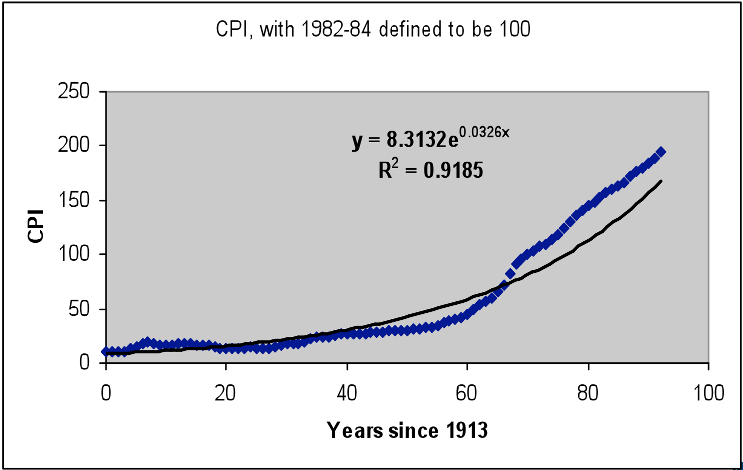

Figure: Cost price index for the last 100 years

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What skill is required to see the equivalence?

- Manipulative skill.

$$\log(Ce^{kt}) = kt + \log(C)$$

- Symbol sense. Understanding the structural similarity between

$$mx + b \quad \text{and} \quad Ca^x$$

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An example from biology: the Michaelis-Menten equation

- v_0 initial velocity of reaction
- $[S]_0$ is initial concentration of substrate
- v_{max}, K_M are constants

$$v_0 = \frac{v_{\text{max}}[S]_0}{K_M + [S]_0}$$

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How do you know if a reaction follows the Michaelis-Menten equation?

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An example from biology: the Michaelis-Menten equation

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How do you know if a reaction follows the Michaelis-Menten equation?

$$\frac{1}{v_0} = \frac{K_M + [S]_0}{v_{\max}[S]_0}$$
$$\frac{1}{v_0} = \frac{K_M}{v_{\max}} \cdot \frac{1}{[S]_0} + \frac{1}{v_{\max}}$$

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An example from physics

$$L_0\sqrt{1-\left(\frac{v}{c}\right)^2}$$

- What value of v makes this equal to L_0?
- What value of v makes it equal to 0?

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An example from finance

$$P(1 + \frac{r}{12})^{12n} = 10{,}000$$

What are the differences between solving this equation for P, r, and n?

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