Symbols and Science

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Institute for Mathematics and Education
The University of Arizona

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Cardano, Sponge Cake, and Notation

Cardano's famous book "Ars Magna" (1545) marked the end of the medieval period in mathematics and triggered the development of modern mathematical notation. The famous formula for the cubic, as presented in the "Ars Magna", resembles more a culinary recipe than a modern formula. The switchover has implications for the teaching of mathematics.

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One view of algebra: quadratic equations

The quadratic formula

A number x satisfies

$$x^2 + bx + c = 0$$

if and only if

$$x = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Factoring

If $r + s = -b$ and $rs = c$, then

$$x^2 + bx + c = (x - r)(x - s)$$

and

$$x^2 + bx + c = 0 \qquad \text{if and only if} \qquad x = r \quad \text{or} \quad x = s.$$

Another view of algebra: understanding functions

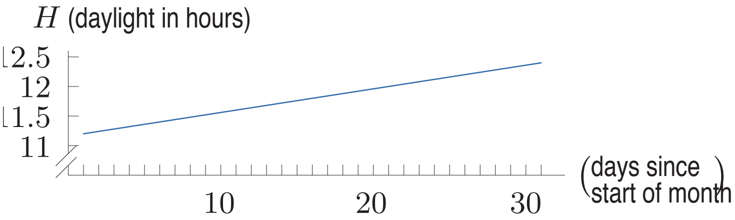

Figure: Sunlight in Madrid

The graph shows the number of hours of daylight in Madrid for one month.

- Why does the graph look linear?
- Estimate and interpret the slope of the line.
- What month does the graph show?

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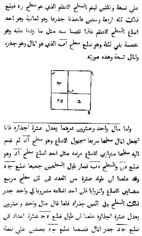
What symbolic skills do modern students need?

- Manipulative skills?
- Graphical and numerical understanding?
- Symbol sense?

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830



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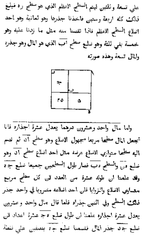

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ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
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خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
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$$x^2 + 10x = 39$$

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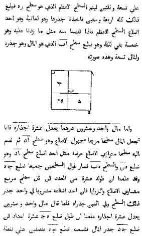

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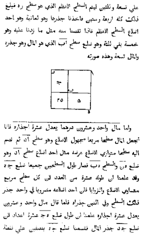

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$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

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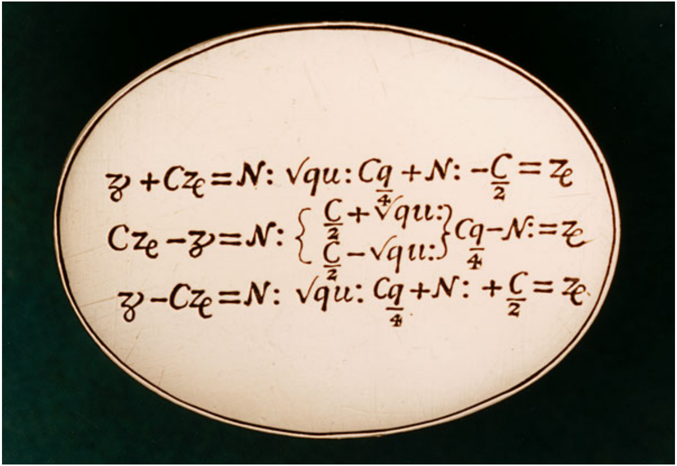

From the Oxford Museum of History of Science (Stephen Johnston, photo Bluebridge Farm Studio)

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What is going on here?

$$z + Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : - \frac{C}{2} = r$$

$$Cr - z = N : \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \frac{Cq}{4} - N : \; = r$$

$$z - Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : + \frac{C}{2} = r$$

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What is going on here?

$$x^2 + Cx = N : \sqrt{}qu : \frac{Cq}{4} + N : \; - \frac{C}{2} = x$$

$$Cx - x^2 = N : \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{}qu : \\ \frac{C}{2} - \sqrt{}qu : \end{array} \right\} \frac{Cq}{4} - N : \; = x$$

$$x^2 - Cx = N : \sqrt{}qu : \frac{Cq}{4} + N : \; + \frac{C}{2} = x$$

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What is going on here?

$$x^2 + Cx = N, \qquad \sqrt{}qu : \frac{C^2}{4} + N : -\frac{C}{2} = x$$

$$Cx - x^2 = N, \qquad \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{}qu : \\ \frac{C}{2} - \sqrt{}qu : \end{array} \right\} \frac{C^2}{4} - N : = x$$

$$x^2 - Cx = N, \qquad \sqrt{}qu : \frac{C^2}{4} + N : +\frac{C}{2} = x$$

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What is going on here?

$$
\begin{aligned}
x^2 + Cx &= N, & \sqrt{qu} : \frac{C^2}{4} + N : -\frac{C}{2} &= x \\
Cx - x^2 &= N, & \frac{C}{2} \pm \sqrt{qu} : \frac{C^2}{4} - N : &= x \\
x^2 - Cx &= N, & \sqrt{qu} : \frac{C^2}{4} + N : +\frac{C}{2} &= x
\end{aligned}
$$

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What is going on here?

$$x^2 + Cx = N, \qquad \sqrt{\frac{C^2}{4} + N} - \frac{C}{2} = x$$

$$Cx - x^2 = N, \qquad \frac{C}{2} \pm \sqrt{\frac{C^2}{4} - N} = x$$

$$x^2 - Cx = N, \qquad \sqrt{\frac{C^2}{4} + N} + \frac{C}{2} = x$$

Viete's formulae and the quadratic formula

If

$$x^2 + bx + c = 0$$

then let

$$r = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

▸ Exercise

If

$$x^2 + bx + c = 0$$

then let

$$r = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4c}}{2}$$

Exercise

Give an explanation, purely in terms of the structure of the expressions, of why these two numbers satisfy

$$r + s = -b \quad \text{and} \quad rs = c.$$

Answer Skip

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Viete's formulae and the quadratic formula

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Answer

When you add r and s, the plus and minus signs cancel.

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If

$$x^2 + bx + c = 0$$

then let

$$r = \frac{-\,b \qquad\qquad\qquad}{2} \quad \text{and} \quad s = \frac{-\,b \qquad\qquad\qquad}{2}$$

Answer

When you add r and s, the plus and minus signs cancel.

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Viete's formulae and the quadratic formula

If

$$x^2 + bx + c = 0$$

then let

$$r = \frac{\boxed{} + \boxed{}}{2} \quad \text{and} \quad s = \frac{\boxed{} - \boxed{}}{2}$$

Answer

When you add r and s, the plus and minus signs cancel.

When you multiply r and s, you get the difference of two squares in the numerator,

$$(-b)^2 - (\sqrt{b^2 - 4c})^2 = b^2 - (b^2 - 4c) = 4c.$$

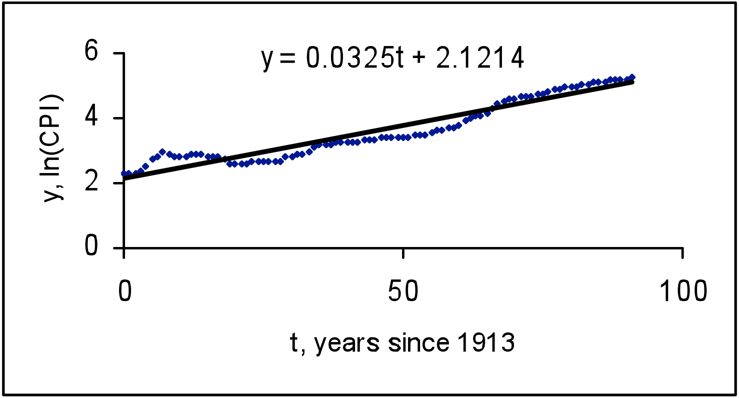

Figure: Cost price index for the last 100 years

Another view of the same data

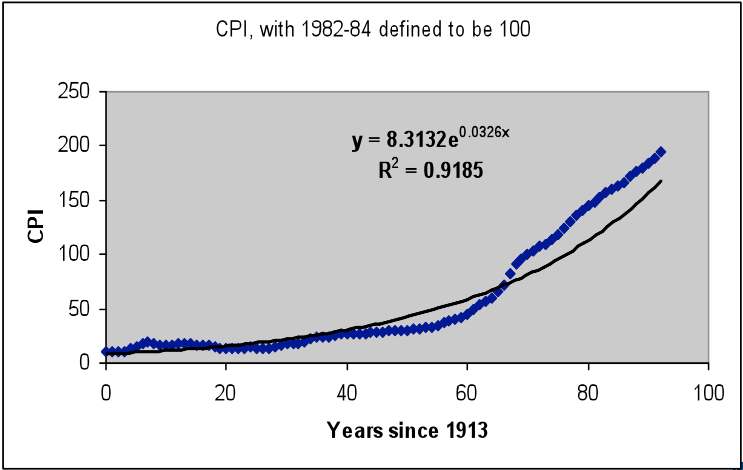

Figure: Cost price index for the last 100 years

What skill is required to see the equivalence?

- Manipulative skill.

$$\log(Ce^{kt}) = kt + \log(C)$$

- Symbol sense. Understanding the structural similarity between

$$mx + b \quad \text{and} \quad Ca^x$$

An example from biology: the Michaelis-Menten equation

- v_0 initial velocity of reaction
- $[S]_0$ is initial concentration of substrate
- v_{max}, K_M are constants

$$v_0 = \frac{v_{\mathrm{max}}[S]_0}{K_M + [S]_0}$$

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An example from biology: the Michaelis-Menten equation

- v_0 initial velocity of reaction
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How do you know if a reaction follows the Michaelis-Menten equation?

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An example from biology: the Michaelis-Menten equation

- v_0 initial velocity of reaction
- $[S]_0$ is initial concentration of substrate
- $v_{\max}$, K_M are constants

$$v_0 = \frac{v_{\max}[S]_0}{K_M + [S]_0}$$

How do you know if a reaction follows the Michaelis-Menten equation?

$$\frac{1}{v_0} = \frac{K_M + [S]_0}{v_{\max}[S]_0}$$
$$\frac{1}{v_0} = \frac{K_M}{v_{\max}} \cdot \frac{1}{[S]_0} + \frac{1}{v_{\max}}$$

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An example from physics

$$L_0\sqrt{1-\left(\frac{v}{c}\right)^2}$$

- What value of v makes this equal to L_0?
- What value of v makes it equal to 0?

An example from finance

$$P(1 + \frac{r}{12})^{12n} = 10{,}000$$

What are the differences between solving this equation for P, r, and n?

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