Creating Mindful Manipulators

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Question: What is the task of a mathematician?
Answer: The task of a mathematician is to increase human understanding of mathematics.

- This encompasses both research and education.
- Increasing human understanding of mathematics requires:
  - thinking about the mathematics
  - thinking about human understanding.
- Simultaneous attention to both is necessary . . . but often forgotten.

What is a polynomial?

- We ask students to factor and expand them, so they are expressions.
- We ask students to graph them, so they are functions.
- We ask students to find their roots, so they are equations.!?
- Students often don't distinguish between

$$p(x), \quad y = p(x), \quad \text{and} \quad p(x) = 0.$$

- Mathematical answer: Neither an expression nor a function. A polynomial is an element of a ring; it can be represented by many different expressions, but it is not a function.
- Pedagogical answer: Both an expression and a function. A polynomial can be regarded as either an expression or a function, depending on the context and the grade level.
- *Our* task (mathematicians and educators) is to reconcile these two answers.

- A mathematics curriculum has both *logical* structure and a *pedagogical structure*.
- Analysis of the first is a mathematical task.
- Analysis of the second is a task of mathematics education.
- Performing these tasks separately is like choosing the plants in a garden without knowing how they grow, then watering and feeding without knowing anything about them.
- We need analysis of the curriculum that is simultaneously mathematical and pedagogical.

An algebra problem

The following question was given to 8th grade algebra students. You are simplifying

$$7 - 2(3 - 8x).$$

Which of the expressions is a correct next step?

	Correct	Not correct
$5(3 - 8x)$		
$7 - 2(-5x)$		
$7 - 6 - 16x$		
$7 - 6 + 16x$		

For each expression, explain why you made the choice you did.

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$$7 - 2(3 - 8x)$$

- $7 - 2(-5x)$ is correct. Always do the parentheses first!
- $7 - 6 + 16x$ is not correct. Where did the 16 come from?
- Can't use the distributive law because of the negative 2.
- $7 - 6 + 16x$ is correct. You have to double multiply first.

Question for undergraduate education

What mathematical skills do we need to give teachers in our undergraduate courses in order to deal with these answers?

Sample activity from algebra course

Problem

The expression

$$0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right)$$

is the contribution to a student's final score from three test scores. What is a different way of writing this? Which way should a student use in order to

- calculate the total test contribution to their final grade
- calculate the effect of getting 10 more points on test 2

Responses

$$0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right), \ 0.2t_1 + 0.2t_2 + 0.2t_3, \ \frac{t_1}{5} + \frac{t_2}{5} + \frac{t_3}{5}, \dots$$

$$(1)\ 0.6 \left(\frac{t_1 + t_2 + t_3}{3} \right) \qquad (2)\ 0.2t_1 + 0.2t_2 + 0.2t_3$$

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Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

Student B: (1) and (2) are obviously the same!

Student A: How you can see that just by looking at them?

Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distributed the 0.2.

Instructor: How do you know you can move the 3 over? What rule says you can do that?

Student B: Isn't it because you only have division and multiplication, so it's the commutative law?

Instructor: But division isn't commutative.

Student C: But you can write division as multiplication. Just write it as multiplication by 1/3.

Student A: Oh yeah! [Discussion shifts to associative law.]