Excavating School Mathematics

William McCallum
The University of Arizona

ICM 2010

A proposal to eliminate quadratic equations

BBC NEWS WORLD EDITION 21 April 2003

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"pupils should be numerate . . . but numeracy can be divorced from mathematics. . . . How often do the majority of people need or use mathematical concepts once they have left school?"

[He advocated] allowing them to drop advanced concepts such as quadratic equations and trigonometry at the age of 14.

The proposal is debated in parliament

The United Kingdom Parliament

26 June 2003

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"A quadratic equation is not like a bleak room, devoid of furniture, in which one is asked to squat. It is a door to a room full of the unparalleled riches of human intellectual achievement. If you do not go through that door . . . much that passes for human wisdom will be forever denied you."

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"Oh dear. I would like to have support from elsewhere as well."

The minister replies

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The minister replies

The UNITED KINGDOM PARLIAMENT

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- The 16-year-old thought that quadratic equations were logical and fairly straightforward because 'you substitute stuff into a formula'. . . .
- The head of maths said that quadratic equations formed an important step in students' ability to solve equations, . . .
- The engineer said that he did not use quadratic equations now, but had in the past . . ."

‘you substitute stuff into a formula’

SOLVING QUADRATIC EQUATIONS

This program solves Quadratic Equations. Enter the coefficients in appropiate boxes and click Solve. It will show th
results in boxes Root1 and Root2.

Enter the Coefficient of X^2 here 1

Enter the Coefficient of X here 1

Enter the Constant here 1

SOLVE

Results:

Root1	-0.5 + sqrt(--0.75)	Or	-0.5+0.8660254037844386i
Root2	-0.5 - sqrt(--0.75)	Or	-0.5-0.8660254037844386i

What do these artefacts tell us?

- School mathematics in the view of many is divorced from reality: "how often do the majority of people need or use mathematical concepts once they have left school?"
- Even those who defend a particular topic cannot always find the words to do so; the topic has become an eroded shell, leading to endless dry exercises ("you substitute stuff into a formula"), and the *reductio ad absurdum* of the online solver.
- A rational decision on whether or not to include quadratic equations in the curriculum, and for whom, should at least depend on a clear understanding of the "unparalleled riches of intellectual achievement" that inhere in it. This is an area where the mathematician can contribute.

Brahmagupta, Brahmasphuta-siddhanta, 628

Considering an equation that in modern notation we would write as $ax^2 + bx = c$, Brahmagupta (628) writes:

> *To the absolute number multiplied by four times the [coefficient of the] square, add the square of the [coefficient of the] middle term; the square root of the same, less the [coefficient of the] middle term, being divided by twice the [coefficient of the] square is the value.*

Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
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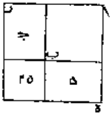

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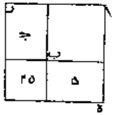

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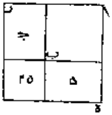

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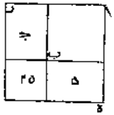

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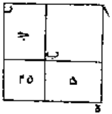

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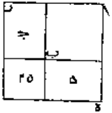

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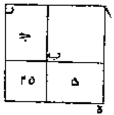

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$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

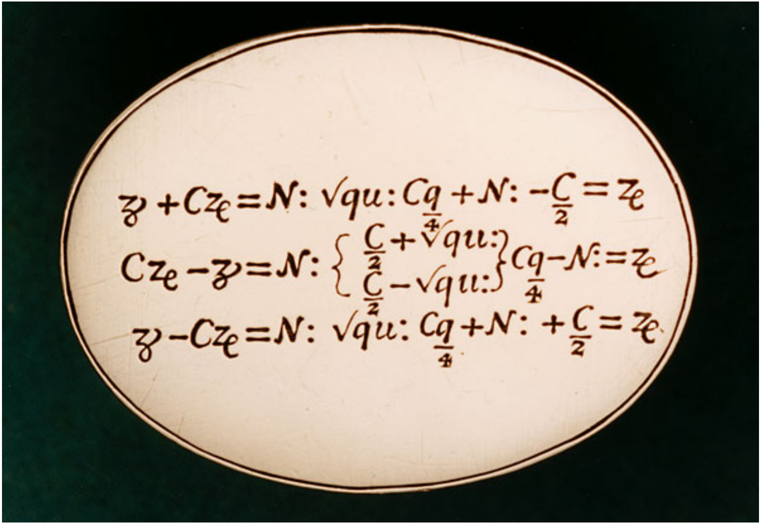

Figure: 17th C English medallion

$$z + Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : \; - \frac{C}{2} = r$$

$$Cr - z = N : \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \frac{Cq}{4} - N : \; = r$$

$$z - Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : \; + \frac{C}{2} = r$$

$$x^2 + Cx = N : \sqrt{qu} : \frac{Cq}{4} + N : \, - \frac{C}{2} = x$$

$$Cx - x^2 = N : \left\{ \begin{array}{c} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \frac{Cq}{4} - N : \, = x$$

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$$Cx - x^2 = N, \qquad \frac{C}{2} \pm \sqrt{}qu : \frac{C^2}{4} - N : = x$$

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$$Cx - x^2 = N, \qquad \frac{C}{2} \pm \sqrt{\frac{C^2}{4} - N} = x$$

$$x^2 - Cx = N, \qquad \sqrt{\frac{C^2}{4} + N} + \frac{C}{2} = x$$

What are the “riches of human intellectual achievement” surrounding the quadratic formula?

- Completing the square: the fact that every quadratic equation can be reduced to an equation of the form

$$(x - p)^2 = q.$$

(and the related fact that all parabolas are similar).

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$$(x - p)^2 = q.$$

  (and the related fact that all parabolas are similar).
- The idea of a number system (rational, real, complex).

What are the “riches of human intellectual achievement” surrounding the quadratic formula?

- Completing the square: the fact that every quadratic equation can be reduced to an equation of the form

$$(x - p)^2 = q.$$

(and the related fact that all parabolas are similar).

- The idea of a number system (rational, real, complex).
- The power of algebraic notation to reveal structure:

$$ax^2 + bx + c = a(x - r)(x - s)$$

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$r + s = -\frac{b}{a} \quad \text{and} \quad rs = \frac{c}{a}.$$

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