

Excavating School Mathematics

William McCallum
The University of Arizona

ICM 2010

A proposal to eliminate quadratic equations



21 April 2003

Terry Bladen, president of the National Association of Schoolmasters Union of Women Teachers:



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“pupils should be numerate . . . but numeracy can be divorced from mathematics. . . How often do the majority of people need or use mathematical concepts once they have left school?”

[He advocated] allowing them to drop advanced concepts such as **quadratic equations** and trigonometry at the age of 14.

The proposal is debated in parliament



The UNITED KINGDOM PARLIAMENT

26 June 2003

Tony McWalter, Labour MP



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“A quadratic equation is not like a bleak room, devoid of furniture, in which one is asked to squat. It is a door to a room full of the unparalleled riches of human intellectual achievement. If you do not go through that door . . . much that passes for human wisdom will be forever denied you.”

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“Oh dear. I would like to have support from elsewhere as well.”

The minister replies



The UNITED KINGDOM PARLIAMENT

26 June 2003

Alan Johnson, Minister



The minister replies



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“ In preparing for this debate, the DFES conducted a straw poll involving a 16-year-old who had just sat maths GCSE, a head of maths and an experienced chemical engineer.”

The minister replies



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- ▶ The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .

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- ▶ The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .
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- ▶ The head of maths said that quadratic equations formed an important step in students’ ability to solve equations, . . .
- ▶ The engineer said that he did not use quadratic equations now, but had in the past . . .”

'you substitute stuff into a formula'

SOLVING QUADRATIC EQUATIONS

This program solves Quadratic Equations. Enter the coefficients in appropriate boxes and click Solve. It will show the results in boxes Root1 and Root2.

Enter the Coefficient of X^2 here

Enter the Coefficient of X here

Enter the Constant here

SOLVE

Results:

Root1 Or

Root2 Or

What do these artefacts tell us?

- ▶ School mathematics in the view of many is divorced from reality: “how often do the majority of people need or use mathematical concepts once they have left school?”
- ▶ Even those who defend a particular topic cannot always find the words to do so; the topic has become an eroded shell, leading to endless dry exercises (“you substitute stuff into a formula”), and the *reductio ad absurdum* of the online solver.
- ▶ A rational decision on whether or not to include quadratic equations in the curriculum, and for whom, should at least depend on a clear understanding of the “unparalleled riches of intellectual achievement” that inhere in it. This is an area where the mathematician can contribute.

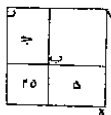
Brahmagupta, Brahmasphuta-siddhanta, 628

Considering an equation that in modern notation we would write as $ax^2 + bx = c$, Brahmagupta (628) writes:

To the absolute number multiplied by four times the [coefficient of the] square, add the square of the [coefficient of the] middle term; the square root of the same, less the [coefficient of the] middle term, being divided by twice the [coefficient of the] square is the value.

Al-Khwarismi, Hisab al-jabr w' al-muqabala, 830

علي تسعة وثلاثين ليم السطح الأعظم الذي هو سطح ره فيلج
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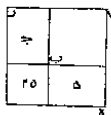


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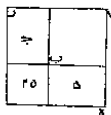


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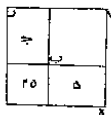
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$$x^2 + 10x = 39$$

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علي تسعة وثلاثين ليمس السطح الأعظم الذي هو سطح ره فيلج ذلكت كله أربعة وستين فإخذنا جذريها وهو لعمانية وهو احد اضلاع السطح الأعظم فاذا تقصنا منه مثل ما زدنا عليه وهو خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره والمال تسعة وهذه صورته



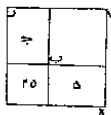
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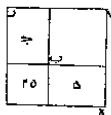
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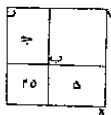
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$$x + 5 = 8$$

$$x = 3$$

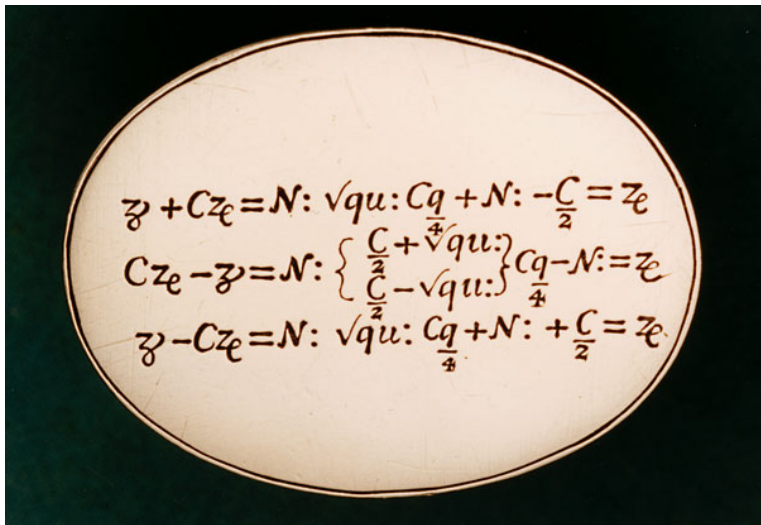


Figure: 17th C English medallion

$$z + Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : -\frac{C}{2} = r$$

$$Cr - z = N : \left\{ \begin{array}{l} \frac{C}{2} + \sqrt{qu} : \\ \frac{C}{2} - \sqrt{qu} : \end{array} \right\} \frac{Cq}{4} - N : = r$$

$$z - Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : +\frac{C}{2} = r$$

$$x^2 + Cx = N : \sqrt{qu} : \frac{Cq}{4} + N : -\frac{C}{2} = x$$

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$$x^2 - Cx = N, \quad \sqrt{\frac{C^2}{4} + N} + \frac{C}{2} = x$$

What are the “riches of human intellectual achievement” surrounding the quadratic formula?

- ▶ Completing the square: the fact that every quadratic equation can be reduced to an equation of the form

$$(x - p)^2 = q.$$

(and the related fact that all parabolas are similar).

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$$(x - p)^2 = q.$$

(and the related fact that all parabolas are similar).

- ▶ The idea of a number system (rational, real, complex).

What are the “riches of human intellectual achievement” surrounding the quadratic formula?

- ▶ Completing the square: the fact that every quadratic equation can be reduced to an equation of the form

$$(x - p)^2 = q.$$

(and the related fact that all parabolas are similar).

- ▶ The idea of a number system (rational, real, complex).
- ▶ The power of algebraic notation to reveal structure:

$$ax^2 + bx + c = a(x - r)(x - s)$$

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$r + s = -\frac{b}{a} \quad \text{and} \quad rs = \frac{c}{a}.$$

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