A Capstone Course for Secondary Teachers

William McCallum

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Where does this solution go wrong?

$$x^2 - 3x - 4 = 0 \quad (1)$$

$$x^2 - 3x = 4 \quad (2)$$

$$x(x - 3) = 4 \quad (3)$$

$$x = 2, \qquad x - 3 = 2 \quad (4)$$

$$x = 2, \qquad x = 5 \quad (5)$$

Where does this solution go wrong?

$$\begin{aligned}
\text{If } x^2 - 3x - 4 &= 0 && (1)\\
\text{then } x^2 - 3x &= 4 && (2)\\
\text{Therefore } x(x-3) &= 4 && (3)\\
\text{and hence } x = 2, &\quad \text{and} \quad x - 3 = 2 && (4)\\
\text{so } x = 2, &\quad \text{and} \quad x = 5 && (5)
\end{aligned}$$

Solving equations and proving theorems

If x is a number such that

$$x^2 - 3x - 4 = 0$$

then

$$(x-4)(x+1) = 0$$

because

$$x^2 - 3x - 4 = (x-4)(x+1)$$

for all x (by the distributive law). If the product of two numbers is zero, then one of them must be zero, so either

$$x - 4 = 0 \quad \text{or} \quad x + 1 = 0, \quad \text{so} \quad x = 4 \quad \text{or} \quad x = -1.$$

Conversely, if $x = 4$ or $x = -1$, then

$$x^2 - 3x - 4 = 0.$$

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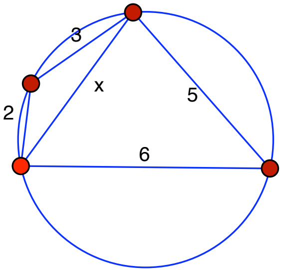
Factoring quadratic expressions

In your high school class, you explain that to factor $x^2 - 3x - 4$, you consider the factorizations of -4, which are -4×1, 4×-1, and 2×-2, and pick the one whose factors add up to -3, namely -4×1, and write $x^2 - 3x - 4 = (x - 4)(x + 1)$. So far so good. Then a student asks why you consider only integer factorizations? Why not $-4 = (-1/2) \times 8$, or $-4 = (-3/2) \times (8/3)$?

1. Why don't we consider factorizations like the ones above?
2. What would be a complete mathematical answer to this question?
3. How do you respond to the student?

Looking at expressions

Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3 (Askey).



Looking at expressions

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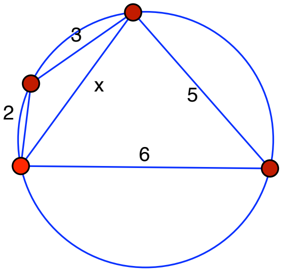
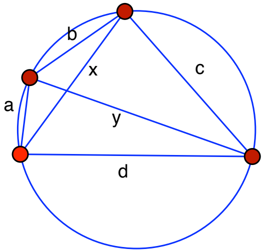


$$
\begin{aligned}
x^2 &= 4 + 9 - 2 \cdot 6 \cos\theta \\
&= 25 + 36 + 2 \cdot 30 \cos\theta
\end{aligned}
$$

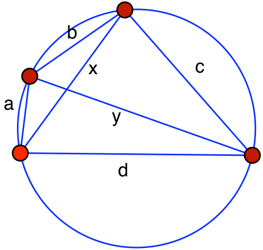
Demo

Ptolemy's theorem



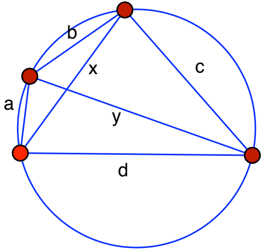
$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd}$$

Ptolemy's theorem



$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} = \frac{(ac + bd)(ad + bc)}{ab + cd}$$

Ptolemy's theorem



$$x^2 = \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} = \frac{(ac+bd)(ad+bc)}{ab+cd}$$

Theorem (Ptolemy's Theorem)

$$xy = ac + bd$$

Viete's formulas and the quadratic formula

If

$$ax^2 + bx + c = 0$$

then let

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

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Exercise

Give an explanation, purely in terms of the structure of the expressions, of why these two numbers satisfy

$$r + s = -\frac{b}{a} \quad \text{and} \quad rs = \frac{c}{a}.$$

Answer

Viete's formulas and the quadratic formula

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Answer

When you add r and s, the plus and minus signs cancel.

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Viete's formulas and the quadratic formula

If

$$ax^2 + bx + c = 0$$

then let

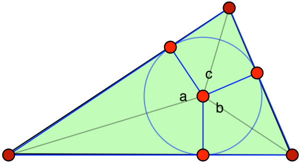
$$r = \frac{\Box + \boxed{}}{2a} \quad \text{and} \quad s = \frac{\Box - \boxed{}}{2a}$$

Answer

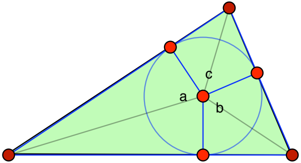
When you add r and s, the plus and minus signs cancel. When you multiply r and s, you get the difference of two squares in the numerator,

$$(-b)^2 - (\sqrt{b^2 - ac})^2 = b^2 - (b^2 - 4c) = 4c.$$

Triangles of fixed area and perimeter



Triangles of fixed area and perimeter

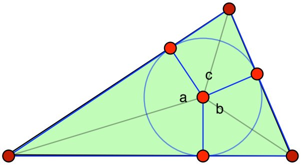


Demo

$$\text{area} = r\left(\frac{\text{perimeter}}{2}\right)$$

$$\text{perimeter} = 2r\left(\tan\frac{a}{2} + \tan\frac{b}{2} + \tan\frac{c}{2}\right)$$

Triangles of fixed area and perimeter



$$\text{area} = r\left(\frac{\text{perimeter}}{2}\right)$$

$$\text{perimeter} = 2r(\tan\frac{a}{2} + \tan\frac{b}{2} + \tan\frac{c}{2})$$

Demo

Let $x = \tan(a/2)$, $y = \tan(b/2)$, then triangles with fixed perimeter and area are parameterized by the cubic curve

$$x + y - \frac{x+y}{1-xy} = k.$$

Graph

Building a course around intracurricular activities

- High school mathematics is full of interesting questions with dense ramifications.
- These questions arise naturally starting from simple tasks of high school mathematics.
- You can design an entire curriculum just by looking at high school mathematics with a mathematical eye and wondering where it leads you.
- Problem of coherence, students getting it.