Proof problems for final exam (may add more later). Some of these problems are theorems you learned. You cannot simply quote the theorems as proofs. You must produce the proofs of the statements.

1 Let A, B be $n \times n$ matrices. Prove that AB is invertible if and only if A, B are invertible.

2 Let A be $m \times n$ matrix. Prove that every vector in null(A) is orthogonal to every vector in row(A) and any vector in null(A^T) is orthogonal to every vector in col(A).

3 Suppose $A \sim B$, show that A, B have the same characteristic polynomial and hence the same set of eigenvalues with the same algebraic multiplicities for each eigenvalue.

4 Show that if A, B are invertible. then $AB \sim BA$.

5 Prove the Cauchy-Schwarz inequality for $\mathbb{R}^n$: $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||||\mathbf{v}||$.

6 Show that if $\{v_1, v_2, \cdots, v_k\}$ is a nonzero orthogonal set in $\mathbb{R}^n$, then it is also a linearly independent set.

7 Consider a linear transformation $T : V \to V$. Suppose $[T]_B$ and $[T]_C$ are matrix representations for T under bases B and C for V respectively. Show that $[T]_B \sim [T]_C$.

8 Let V be a vector space with basis $\{v_1, v_2, \cdots, v_n\}$. Show that for every vector $u \in V$, there exists unique $c_1, \cdots, c_n \in F$ such that $u = c_1v_1 + c_2v_2 + \cdots + c_nv_n$.

9 Prove that the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the column space of its standard matrix representation, $[T]$.

10 Let W be a subspace of V. Prove that $W^\perp$ is also a subspace of V, and show that $W, W^\perp$ form a decomposition of V, i.e. $W + W^\perp = V$ and $W \cap W^\perp = \{0\}$.