Practice Test 3

For full credit, show all work. You can use your calculator for computational purposes. But your work on paper must be transparent enough that I understand your answer without a calculator. No credit will be rewarded if I cannot understand how you obtain a numerical answer.

1 Define: *Note: these are just samples.*

a) Eigenvalues and eigenvectors for a matrix $A$.

b) Similarity transformation.

c) General vector space $V$.

2 Consider the matrix $A = \begin{pmatrix} 2 & 1 & -4 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$.

a) Using elementary row operations and row echelon form of $A$, compute the determinant of $A$.

b) Find the eigenvalues for $A$ and compute the respective eigenspace. Determine the algebraic and geometric multiplicities for each eigenvalue.

3a) Consider the matrix $A = \begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. Find the similarity transformation $P$ that diagonalizes $A$. Write down the diagonalization of $A$.

b) Consider the system of linear equation $x' = Ax$ with $A$ as defined in part (a). Find the particular solution that satisfies the initial condition $x_0 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

4 Consider the equation for linear dynamical system $x_{k+1} = Ax_k$ where $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Factorize $A$ into $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ by computing $r$ and $\theta$. Characterize the zero fixed point. Explain.

5a) Show that $B = \{1 - x, 1 + x^2, x - x^2\}$ forms a basis for $P_2(\mathbb{R})$.

b) Let $C = \{1, 1 + x, 1 + x + x^2\}$ be another basis for $P_2(\mathbb{R})$. Find the change of basis matrix from $B$ to $C$, using the standard basis as an intermediate basis.
6 Prove the following statements:

a) Prove that \( \det(kA - \lambda I) = k^n \det(A - \frac{\lambda}{k} I) \).

b) Let \( A \) and \( B \) be \( n \times n \) matrices, each with \( n \) distinct eigenvalues. Prove that \( A \) and \( B \) have the same eigenvectors if and only if \( AB = BA \).