Practice Test 3

For full credit, show all work. You can use your calculator for computational purposes. But your work on paper must be transparent enough that I understand your answer without a calculator. No credit will be rewarded if I cannot understand how you obtain a numerical answer.

1 Define: *Note: these are just samples.*

a) Eigenvalues and eigenvectors for a matrix A.

b) Similarity transformation.

c) General vector space V.

2 Consider the matrix $A = \begin{pmatrix} 2 & 1 & -4 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$.

a) Using elementary row operations and row echelon form of A, compute the determinant of A.

b) Find the eigenvalues for A and compute the respective eigenspace. Determine the algebraic and geometric multiplicities for each eigenvalue.

3a) Consider the matrix $A = \begin{pmatrix} -1 & 0 & 2 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. Find the similarity transformation P that diagonalizes A. Write down the diagonalization of A.

b) Consider the system of linear equation $\mathbf{x}' = A\mathbf{x}$ with A as defined in part (a). Find the particular solution that satisfies the initial condition $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

4 Consider the equation for linear dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ where $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Factorize A into $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ by computing r and θ. Characterize the zero fixed point. Explain.

5a) Show that $\mathcal{B} = \{1 - x, 1 + x^2, x - x^2\}$ forms a basis for $P_2(\mathbb{R})$.

b) Let $\mathcal{C} = \{1, 1 + x, 1 + x + x^2\}$ be another basis for $P_2(\mathbb{R})$. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, using the standard basis as an intermediate basis.

6 Prove the following statements:

a) Prove that $\det(kA - \lambda I) = k^n \det\left(A - \frac{\lambda}{k} I\right)$.

b) Let A and B be $n \times n$ matrices, each with n distinct eigenvalues. Prove that A and B have the same eigenvectors if and only if $AB = BA$.