

# Weak Turbulent Kolmogorov Spectrum for Surface Gravity Waves

A. I. Dyachenko,<sup>1,\*</sup> A. O. Korotkevich,<sup>1,†</sup> and V. E. Zakharov<sup>1,2,‡</sup>

<sup>1</sup>*Landau Institute for Theoretical Physics,  
2, Kosygin Str., Moscow, 119334, Russian Federation*

<sup>2</sup>*University of Arizona,  
Tucson, AZ 85721, USA*

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## Abstract

We study the long-time evolution of gravity waves on deep water excited by the stochastic external force concentrated in moderately small wave numbers. We numerically implement the primitive Euler equations for the potential flow of an ideal fluid with free surface written in canonical variables, using expansion of the Hamiltonian in powers of nonlinearity of up to fourth order terms.

We show that due to nonlinear interaction processes a stationary energy spectrum close to  $|k| \sim k^{-7/2}$  is formed. The observed spectrum can be interpreted as a weak-turbulent Kolmogorov spectrum for a direct cascade of energy.

Kolmogorov was born in 1903. Now, in the year of his centenary, his greatness is obvious not only for pure and applied mathematicians. All physicists appreciate his pioneering works on powerlike cascade spectra in turbulence of the incompressible fluid [1]. It is obvious now that cascade processes, similar to the Kolmogorov cascade of energy, play a very important role in many different fields of physics, such as nonlinear optics [2], plasma physics [3], hydrodynamics of superfluid He<sup>4</sup>, and so forth.

In all these cases the physical situations are similar. There is an ensemble of slowly decaying, weakly nonlinear waves in a medium with dispersion. Such systems have to be described statistically. However this is not traditional statistical mechanics, because the ensembles are very far from the thermodynamic equilibrium. Nevertheless, one can develop a systematic approach for statistical study of weakly nonlinear waves. This is the theory of weak (or wave) turbulence [4]. The main tools here are the kinetic equations for squared wave amplitudes. These equations describe the nonlinear resonant interaction processes, taking place in the wave systems. Like in the turbulence in incompressible fluid, these processes lead to the formation of constants of motion cascades (energy, wave action, momentum etc.) along the  $k$ -space. In isotropic systems it might be either a direct cascade of energy from small to large wave numbers or an inverse cascade of wave action to small wave numbers [5]. In an anisotropic system the situation could be much more complicated [6].

The brilliant conjecture of Kolmogorov still is a hypothesis, supported by ample experimental evidence. On the contrary, the existence of powerlike Kolmogorov spectra, describing cascades in weak turbulence, is a rigorous mathematical fact. These spectra are the exact solutions of the stationary homogeneous kinetic equation, completely different from the thermodynamic Rayley-Jeans solutions.

Nevertheless, the case is not closed. The weak turbulent theory itself is based on some assumptions, like phase stochasticity and the absence of coherent structures. This is the reason why justification of weak turbulent theory is an urgent and important problem.

This justification can be done by a direct numerical solution of the primitive dynamic equation describing the wave ensemble. In pioneering works by Majda, McLaughlin and Tabak [7] it was done for the 1-D wave system. The results obtained by these authors are not easily interpreted. In some cases they demonstrate Kolmogorov-type spectra, in other cases — power spectra with essentially different exponents.

In article [8] deviation from weak turbulent theory was explained by the role of coherent

structures (solitons, quasisolitons and collapses). If a 1-D system is free from coherent structures, weak-turbulent spectra are observed with a good deal of evidence [9, 10]

In spite of their heuristic value, the 1-D models so far developed have no direct physical application. Real physical systems, where wave turbulence is realized, are at least 2-dimensional. The most natural and important examples are capillary and gravity waves on deep water. Weak-turbulent theory of capillary waves was developed by Zakharov and Filonenko in 1967 [11], who found that the correlation function of evolution has to be  $\langle |\eta_{\vec{k}}|^2 \rangle \sim k^{-19/4}$ . This result was supported by laboratory experiments, performed independently by three groups (in UCLA [12], Niels Bohr Institute [13] and the Solid State Physics Institute in Chernogolovka, Russia [14, 15]). The spectrum  $k^{-19/4}$  was obtained by a direct numerical simulation of Euler equation for incompressible fluid with free surface by Pushkarev and Zakharov [16–18].

The most interesting example of 2-D wave ensembles demonstrating weak-turbulent cascades is a system of gravity waves on the surface of deep water. We sure that the weak-turbulent theory of these waves is key to understanding the processes in a wind-driven sea. However, we do not concentrate on this point in our article.

Our initial goal was to reproduce (and emulate), for gravity waves, the work which was done by Pushkarev and Zakharov [16] for capillary waves. One has to expect that this is a more difficult problem, because the leading process in capillary waves is a three-wave interaction, while for gravity waves the lowest order process is four-wave interaction.

Attempts to perform direct numerical simulations of potential flow in an ideal fluid with a free surface were made by several authors [19]. Only in one article authors paid interest to Kolmogorov-type weak-turbulent spectra [20]. Authors of this paper observed the formation of Kolmogorov tails in the process of time evolution of an artificially cut-off JONSWAP energy spectrum [21]. The results of presented article agree with the results of [20] completely, however we would like to stress a difference.

In our work we study a forced turbulence, excited by external sources, posed in moderately low wave numbers. We show that growth of wave energy due to this forcing is arrested by the nonlinear resonant four-wave processes which leads to the formation of powerlike Kolmogorov spectrum in the transparency domain. In this sense our article is a direct numerical confirmation of the weak-turbulent theory for surface gravity waves.

*Theoretical Background* — So we study the potential flow of an ideal inviscid incom-

compressible fluid with the velocity potential  $\phi = \phi(x, y, z; t)$

$$\Delta\phi = 0,$$

in the infinitely deep domain occupied by the fluid. Equations for the boundary conditions at the surface are the following

$$\begin{aligned} & \left. \left( \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x} \frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y} \frac{\partial\eta}{\partial y} \right) \right|_{z=\eta} = \left. \frac{\partial\phi}{\partial z} \right|_{z=\eta}, \\ & \left. \left( \frac{\partial\phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial\phi}{\partial x} \right)^2 + \left( \frac{\partial\phi}{\partial y} \right)^2 + \left( \frac{\partial\phi}{\partial z} \right)^2 \right] \right) \right|_{z=\eta} + \\ & + g\eta = 0. \end{aligned} \quad (1)$$

Here  $\eta(x, y; t)$  is the surface elevation with respect to still water,  $g$  is the gravity acceleration. Equations (1) are Hamiltonian [22] with the canonical variables  $\eta(x, y; t)$  and  $\psi(x, y; t) = \phi(x, y, \eta(x, y; t); t)$

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \quad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \quad (2)$$

where  $H$  is the Hamiltonian of the system

$$\begin{aligned} H &= H_{\text{kin}} + H_{\text{pot}}, \\ H_{\text{kin}} &= \frac{1}{2} \int_{-\infty}^{+\infty} dx dy \int_{-\infty}^{\eta} (\nabla\phi)^2 dz, \\ H_{\text{pot}} &= \frac{1}{2} g \int_{-\infty}^{+\infty} \eta^2 dx dy, \end{aligned}$$

Unfortunately  $H$  cannot be written in the close form as a functional of  $\eta$  and  $\psi$ . However one can limit Hamiltonian by first three terms of powers of  $\eta$

$$\begin{aligned} H &= \frac{1}{2} \int (g\eta^2 + \psi \hat{k}\psi) dx dy + \\ & + \frac{1}{2} \int \eta [|\nabla\psi|^2 - (\hat{k}\psi)^2] dx dy + \\ & + \frac{1}{2} \int \eta(\hat{k}\psi) [\hat{k}(\eta(\hat{k}\psi)) + \eta\Delta\psi] dx dy. \end{aligned} \quad (3)$$

Here  $\hat{k}$  is a linear integral operator ( $\hat{k} = \sqrt{-\Delta}$ ), such that in  $k$ -space it corresponds to multiplication of Fourier harmonics by  $\sqrt{k_x^2 + k_y^2}$ . For gravity waves this reduced Hamiltonian describes four-wave interaction. Than dynamical equations (2) acquire the form

$$\begin{aligned} \dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\ & + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \frac{1}{2}\Delta[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\Delta\psi], \\ \dot{\psi} &= -g\eta - \frac{1}{2} [(\nabla\psi)^2 - (\hat{k}\psi)^2] - \\ & - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi + D_{\vec{r}} + F_{\vec{r}}. \end{aligned} \quad (4)$$

Here  $D_{\vec{k}}$  is the "artificial" damping which simulates viscosity on small scales and  $F_{\vec{k}}$  is the driving term which simulates pumping on large scales (say, due to wind). In the  $k$ -space supports of  $D_{\vec{k}}$  and  $F_{\vec{k}}$  are separated by the inertial interval, where the Kolmogorov-type solution can be recognized.

We study numerically the quasi stationary solution of equations (4). According to the theory of weak turbulence the surface elevation spectrum averaged by ensemble corresponding to the flux of energy from large scales to small scales is

$$\langle |\eta_k|^2 \rangle = \frac{Cg^{1/2}P^{1/3}}{k^{7/2}}. \quad (5)$$

Here  $P$  is the energy flux, and  $C$  is a dimensionless Kolmogorov constant.

*Numerical Simulation* — For numerical integration of (4) we used the following pumping and damping terms which are defined in Fourier space as

$$\begin{aligned} F_{\vec{k}} &= f_k e^{iR_{\vec{k}}(t)}, \\ f_k &= 4F_0 \frac{(k - k_{p1})(k_{p2} - k)}{(k_{p2} - k_{p1})^2}; \\ D_{\vec{k}} &= \gamma_k \psi_{\vec{k}}, \\ \gamma_k &= -\gamma_1, k \leq k_{p1}, \\ \gamma_k &= -\gamma_2(k - k_d)^2, k > k_d. \end{aligned} \quad (6)$$

Here  $R_{\vec{k}}(t)$  is the uniformly distributed random number in the interval  $(0, 2\pi)$  for each  $\vec{k}$ . We have applied an implicit difference scheme that retains the main properties of this system — variational principle and conservation of Hamiltonian in the absence of pumping and damping. The details of this scheme will be published separately.

The equations (4) were numerically simulated in periodic domain  $2\pi \times 2\pi$ . The size of the grid was chosen at  $512 \times 512$  points. Gravity acceleration  $g$  was equal to one. Parameters of the damping and pumping in (6) were the following:  $k_{p1} = 5$ ,  $k_{p2} = 10$ ,  $k_d = 100$ . Thus the inertial interval is equal to a decade.

In the simulations we paid special attention to the problems which could "damage" the calculations. First of all, it is the "bottle neck" phenomenon at the boundary between inertial interval and dissipation region. This effect is very fast, but can be effectively suppressed by a proper choice of damping value  $\gamma_2$  in the case of moderate pumping values  $F_0$ . The second problem is the accumulation of "condensate" in low wave numbers. This mechanism for the case of capillary waves was examined in detail in [23]. Buildup of condensate can

be overcome by simple adaptive damping in the small wave numbers. After some time the system reaches the stationary state, where the balance between pumping and damping takes place. In this state an important parameter is the ratio of nonlinear energy to the linear one  $(H_1 + H_2)/H_0$ .

For example, for the external force  $F_0 = 2 \times 10^{-4}$ ,  $\gamma_1 = 1 \times 10^{-3}$ ,  $\gamma_2 = 665$  the level of nonlinearity was equal to  $(H_1 + H_2)/H_0 \simeq 3 \times 10^{-3}$ . The Hamiltonian as a function of time is shown in Fig. 1.

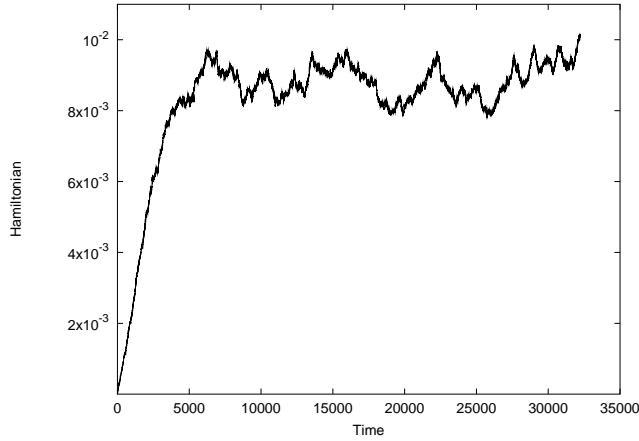


FIG. 1: Hamiltonian as a function of time.

The spectrum of surface elevation appears to be power-like in the essential part of inertial interval, where the influence of pumping and damping was small. This spectrum is shown in Fig. 2.

One can estimate the exponent of the spectrum. It is worth saying that an alternative spectrum was proposed earlier by Phillips [24]. That power-like spectrum is due to wave breaking mechanism and gives us a surface elevation correlator as  $I_k \sim k^{-4}$ . Compensated spectra are shown in Fig. 3. It seems to be the evidence that the Kolmogorov spectrum predicted by weak turbulence theory better fits the results of the numerical experiment.

The quality of the result (closeness to the  $\langle |\eta_k|^2 \rangle \sim k^{-7/2}$ ) crucially depends on the width of the inertial interval. In our previous work [25] similar simulations were performed on the grid  $256 \times 256$ . Weak turbulent spectrum is clearly seen on the grid  $512 \times 512$ , can be divined on the grid  $256 \times 256$ , and is almost invisible on the grid  $128 \times 128$ . This difference is demonstrated in Fig. 4.

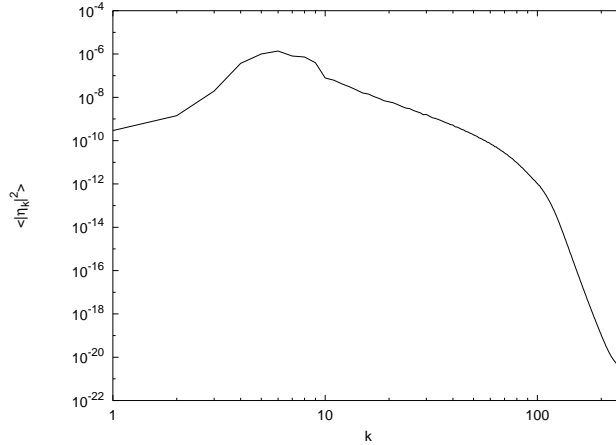


FIG. 2: Surface gravity waves correlator.

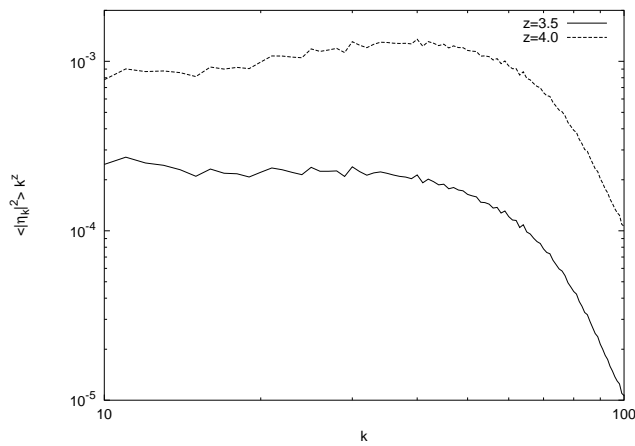


FIG. 3: Compensated waves spectra. One can see, that weak-turbulent theory fits better to results obtained.

In the end we would like to mention that for a different pumping level surface elevation spectra differ only due to the Kolmogorov constant  $C$  in (5), as clearly seen in Fig. 5.

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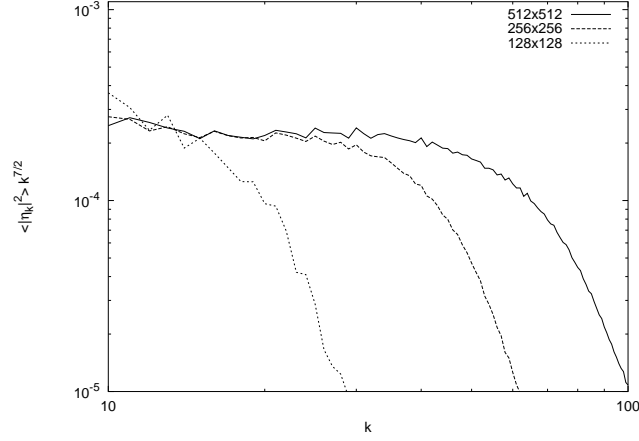


FIG. 4: Broadening of powerlike spectrum interval at the different grids.

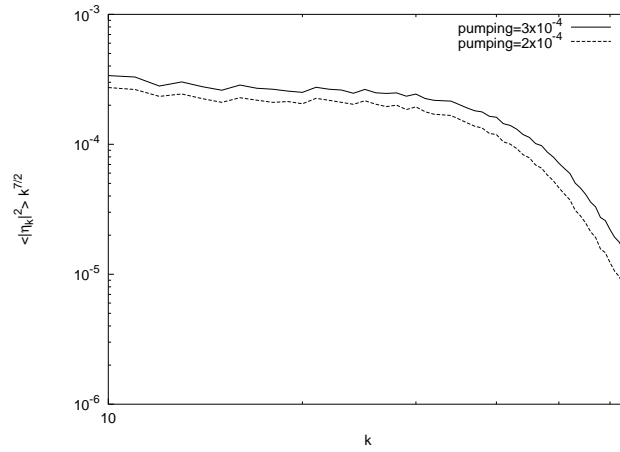


FIG. 5: Spectral inertial interval at the different pumping level.

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\* Electronic address: alexd@landau.ac.ru

† Electronic address: kao@landau.ac.ru

‡ Electronic address: zakharov@math.arizona.edu

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