Integration of the Gauss-Codazzi equation

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1 Introduction

The Bonnet's theorem claims that a surface in $3-D$ eucledean space is defined up to eucledean motions, if components of first and second quadratic forms are known. However, these components can't be chosen by an arbitrary way. Indeed, the first quadratic form defines a metric on the surface, while the second quadratic form determines a field of normals to any point of the surface.

If the components of both forms are known, one can construct a 3-dimensional metrics in a vanishing thin layer nearby the surface. As far as the surface is imbedded in an eucledean space, this metrics is flat, and components of a curvature tensor in a neighbourhood of the surface must be identically zero. This requirment imposes a set of differencial relations to the components of quadratic forms. It is known as a Gauss-Codazzi equation.

In this article we will show that the Gauss-Codazzi equation can be integrated by the use of technique called Inverse Scattering Method. Exactly speaking, we will show that the components of both quadratic forms can be expressed through only one function of two variables. We will call it "master function of a surface".

Actually, the master function defines not a single surface but a whole class of "Combescure equivalent" surfaces. This function is a kernel of a certain linear integral equation. To express the components of quadratic forms in terms of the master function, one has to solve this equation. As soon as it is solved one can construct all Combescure equivalent surfaces in explicit form. The integral equation can be solved efficiently only in some special cases when the kernel is generated and can be presented as a superposition of binary products of the function on one variable. This "solitonic" case certainly deserves a very serious attention, because it makes possible to study efficiently some new classes of surfaces.

Moreover, we can hope that results of this article are of more importance. What we actually do, we reduce a problem of surface classification to a problem of classification of their master functions and and to a more fine classification inside a given Combescure class. All known before classes of surfaces have master functions that satisfy some special conditions. Finding these conditions as well as new conditions defining new special classes of surfaces, is an interesting program for future researches.

This article shows also how deeply the theory of surfaces is connected with the theory of solitons.

2 Formulation of the problem

Let Γ be a surface in R^3. One can introduce coordinates x_1, x_2 on Γ such that first and second quadratic forms are diagonal,

$$\begin{aligned} \Omega_1 &= p^2 dx_1^2 + q^2 dx_2^2, \\ \Omega_2 &= pA dx_1^2 + qB dx_2^2. \end{aligned} \tag{2.1}$$

Here coordinates x_1, x_2 are defined up to trivial transformations $x_1 = x_1(u_1), x_2 = x_2(u_2)$, and we will call two surfaces as "Combescure equivalent", if they have the same A, B but different p, q.

Coefficients of these two quadratic forms Ω_1, Ω_2 cannot be chosen independently. Four functions, p, q, A, B are connected by three nonlinear PDE known as Gauss-Codazzi equations (GCE). To find these equations one should imbed the surface Γ in a special three-orthogonal curvilinear coordinate system in R^3 in vicinity of Γ. The metrics in this system is defined as follow:

$$ds^2 = H_1^2 dx_1^2 + H_2^2 dx_2^2 + dx_3^2, \tag{2.2}$$

where H_1, H_2 are the Lamé coefficients

$$H_1 = p + Ax_3, \quad H_2 = q + Bx_3, \tag{2.3}$$

and the third Lamé coefficient $H_3 = 1$.

The Gauss-Codazzi equations appear from the condition

$$R_{ijlm} = 0, \tag{2.4}$$

where R_{ijlm} is the Riman curvature tensor for metrics (2.2). Equation (2.4) takes a simple form in terms of matrix

$$Q_{ij} = \frac{1}{H_j}\frac{\partial H_1}{\partial x_3}, \quad i \neq j. \tag{2.5}$$

By definition

$$\begin{aligned} Q_{13} &= \frac{1}{H_3}\frac{\partial H_1}{\partial x_3} = A, \quad Q_{23} = \frac{1}{H_3}\frac{\partial H_2}{\partial x_3} = B, \\ Q_{31} &= \frac{1}{H_1}\frac{\partial H_3}{\partial x_1} = 0, \quad Q_{32} = \frac{1}{H_2}\frac{\partial H_3}{\partial x_2} = 0. \end{aligned} \tag{2.6}$$

For the remaining elements of Q_{ij} equation (2.4) reads:

$$\frac{\partial Q_{12}}{\partial x_3} = \frac{\partial Q_{21}}{\partial x_3} = 0, \tag{2.7}$$

$$\frac{\partial Q_{13}}{\partial x_2} = Q_{12}Q_{23}, \quad \frac{\partial Q_{23}}{\partial x_2} = Q_{21}Q_{13}, \tag{2.8}$$

$$\frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{21}}{\partial x_1} + Q_{13}Q_{23} = 0. \tag{2.9}$$

Obviousely, all elements of Q_{ij} do not depend on variable x_3, and one can put

$$W_{12} = \alpha(x_1, x_2), \quad Q_{21} = \beta(x_1, x_2). \tag{2.10}$$

Then the system (2.8-2.9) can be written in a form

$$\begin{aligned} &\frac{\partial \alpha}{\partial x_2} + \frac{\partial \beta}{\partial x_1} + AB = 0, \\ &\frac{\partial A}{\partial x_2} = \alpha B, \quad \frac{\partial B}{\partial x_2} = \beta A. \end{aligned} \tag{2.11}$$

To express α, β in terms of elements of the first quadratic form p, q, one should use definition of Q_{12}, Q_{21}. By definition:

$$\begin{aligned} \frac{\partial}{\partial x_2}(p + Ax_3) &= Q_{12}(q + Bx_3) = \alpha(q + Bx_3), \\ \frac{\partial}{\partial x_1}(q + Bx_3) &= Q_{21}(p + Ax_3) = \beta(p + Ax_3). \end{aligned} \tag{2.12}$$

Putting $x_3 = 0$ in (2.12) we obtain

$$\frac{\partial p}{\partial x_2} = \alpha q, \quad \frac{\partial q}{\partial x_1} = \beta p, \tag{2.13}$$

and get finally

$$\frac{\partial}{\partial x_2}\Big(\frac{1}{q}\frac{\partial p}{\partial x_2}\Big) + \frac{\partial}{\partial x_1}\Big(\frac{1}{p}\frac{\partial q}{\partial x_1}\Big) + AB = 0, \tag{2.14}$$

$$q\frac{\partial A}{\partial x_2} = B\frac{\partial p}{\partial x_2}, \;\; p\frac{\partial B}{\partial x_1} = A\frac{\partial q}{\partial x_1}.$$

The system (2.14) of three equations imposed on four functions A, B, p, q is the Gauss-Cadazzi system. It is suitable to study more simple system of first order equations (2.11). One can mention these equations form a system of three equations imposed on four functions, $A < B < \alpha, \beta$. By solving this system one defines a surface up to Combescure equivalence. To accomplish the solution of Gauss-Codazzi equation one has to solve the linear system (2.13) for components of the first quadratic form. Splitting of the Gauss-Codazzi system (2.14) to more simple systems (2.11), (2.13) was done by Conopelchenko [1].

3 n-Orthogonal coordinate systems

Our approach to solution of the Gauss-Codazzi equations is based on the fact that this is a special degenerated case of Gauss-Lamé equations, describing 3-orthogonal curvilinear coordinate systems in R^3. A method for solution of Gauss-Lamé equations in an eucledean space of arbitrary dimension was presented in [2]. In this article we will give a different, but essencially equivalent method for solution of this problem.

Suppose S is a domain in R^n. How to find all orthogonal curvilinear coordinate systems in S? Let $x = (x_1, \dots, x_n)$ be such coordinates. In this coordinate system the metrix tensor is diagonal,

$$ds^2 = \sum H_i^2 dx_i^2, \tag{3.1}$$

where coefficients $H_i = H_i(x)$, the Lamé coefficients, are subject for determination. They satisfy an heavily overdeterminated system of nonlinear PDE, the Gauss-Lamé equations. These equations read:

$$\frac{\partial Q_{ij}}{\partial x_k} = Q_{ik}Q_{kj}, \;\; i \neq j \neq k, \tag{3.2}$$

$$\frac{\partial Q_{ij}}{\partial x_j}+\frac{\partial Q_{jk}}{\partial x_i}+\sum_{k\neq i,j}Q_{ik}Q_{kj}=0,\ i\neq j. \tag{3.3}$$

Here the same as before

$$Q_{ij}=\frac{1}{H_j}\frac{\partial H_i}{\partial x_j}. \tag{3.4}$$

One can check that equations (3.2)-(3.4) are equivalent to condition

$$R_{ijkl}=0, \tag{3.5}$$

where R_{ijkl} is the Riemann curvature tensor. To solve (3.2)-(3.4) we introduce in R^n a family of projecting operators I_i satisfying the conditions

$$I_i^2=I_i,\ \ I_iI_j=0,\ \ i\neq j, \tag{3.6}$$

and define

$$\Phi=\sum_{i=1}^{n}x_iI_i. \tag{3.7}$$

Let λ is a point on a complex plain C, and $\chi=\chi_{ij}(\lambda,\bar\lambda,x)(i=1,\dots,n;j=1,\dots,n)$ is a matrix-valued function on C, depending also on coordinate x. Suppose that $\chi(\lambda,\bar\lambda,x)$ is a solution of the following non-local $\bar\partial$-problem:

$$\frac{\partial\chi}{\partial\bar\lambda}=\chi\times R=\int\chi(\nu,\bar\nu,x)R(\nu,\bar\nu,\lambda,\bar\lambda,x)d\nu d\bar\nu, \tag{3.8}$$

normalized by the condition $\chi\longrightarrow\delta_{ij}$ at $\lambda\longrightarrow\infty$.

In (3.8)

$$R(\nu,\bar\nu,\lambda,\bar\lambda)=e^{\nu\Phi}Te^{-\lambda\Phi}, \tag{3.9}$$

where $T(\nu,\bar\nu,\lambda,\bar\lambda)$ is a matrix that does not depend on x. We impose on T two restrictions:

$$\begin{aligned}\bar T(\bar\nu,\nu,\bar\lambda,\lambda)&=T(\nu,\bar\nu,\lambda,\bar\lambda),\\ T^{tr}(-\nu,-\bar\nu,-\lambda,-\bar\lambda)&=\frac{\mu}{\lambda}T(\nu,\bar\nu,\lambda,\bar\lambda).\end{aligned} \tag{3.10}$$

The $\bar\partial$-problem (3.8) is equivalent to the integral equation

$$\chi(\lambda,\bar\lambda,x)=\delta_{ij}+\frac{1}{\pi}\int\frac{\chi(\nu,\bar\nu,x)R(\nu,\bar\nu,\mu,\bar\mu,x}{\lambda-\mu}d\nu d\bar\nu d\mu d\bar\mu. \tag{3.11}$$

Suppose that matrix function $T(\nu,\bar{\nu},\lambda,\bar{\lambda})$ satisfying conditions (3.10)-(3.11) is chosen so that equation (3.12) has an unique regular solution. Then χ can be expanded at $\lambda\to\infty$ in the asymptotic series

$$\chi\to 1+\frac{Q}{\lambda}+\frac{P}{\lambda^2}+\cdots, \tag{3.12}$$

where

$$Q=\frac{1}{\pi}\int\chi(\nu,\bar{\nu},x)R(\nu,\bar{\nu},\mu,\bar{\mu},x)d\lambda d\bar{\lambda}d\mu d\bar{\mu}. \tag{3.13}$$

Let us notice that in virtue of (3.10) the function χ satisfies the condition

$$\bar{\chi}(\bar{\nu},\nu,x)=\chi(\nu,\bar{\nu},x) \tag{3.14}$$

and Q is a real matrix function on x.

Using of $\bar{\partial}$-problem (3.8) and equivalent integral equation (3.12) for integration of the Gauss-Lamé system is based on the following two facts:

1. If conditions (3.10), (3.11) are satisfied, matrix $Q_{ij}(x)$ presented by formula (3.14) satisfies systems of equations (3.2), (3.3).

2. The matrix function

$$\phi(\lambda,\bar{\lambda},x)=\chi(\lambda,\bar{\lambda},x)e^{-\lambda\Phi(x)} \tag{3.15}$$

satisfies the linear system

$$\frac{\partial\phi_{ik}}{\partial x_j}=Q_{ij}\phi_{jk},\quad i\neq 0. \tag{3.16}$$

Let $\xi_i(\lambda,\bar{\lambda})=\bar{\xi}_i(\bar{\lambda},\lambda)$ be an arbitrary chosen family of functions on $\lambda,\bar{\lambda}$, and

$$H_i=\int\sum\chi_{ij}(\lambda,\bar{\lambda},x)\xi_i(\lambda,\bar{\lambda})d\lambda d\bar{\lambda}. \tag{3.17}$$

The function $H_i(x)$ satisfies the linear system

$$\frac{\partial H_i}{\partial x_j}=Q_{ij}H_j \tag{3.18}$$

and can be chosen as Lamé coefficient for some n-orthogonal coordinate system. Choosing different sets of $\xi_i(\lambda,\bar{\lambda})$ one will obtain different sets of Lamé coefficients, corresponding to the same matrix Q_{ij}. These sets are the so-called Combescure equivalents. The method of solution of nonlinear equations by use of the $\bar{\partial}$-problem (3.8) is called "dressing method", while function

$T(\nu, \bar{\nu}, \lambda, \bar{\lambda})$ is a "dressing function". One can see that procedure of dressing makes possible to find automatically all sets of Lamé coefficients associated with a given matrix function Q_{ij}.

To prove the statements formulated above one should construct a family of operators $L_{ij} (i \neq j)$, acting on χ by the following way

$$L_{ij}\chi = I_i\Big(\frac{\partial \chi}{\partial x_j} + \lambda\chi I_j - QI_j\chi\Big). \tag{3.19}$$

Here Q is taken from (2.13), and one can easily check that $L_{ij}\chi$ are solutions of $\bar{\partial}$-problem

$$\frac{\partial}{\partial\bar{\lambda}} L_{ij}\chi = L_{ij}\chi \times R$$

with zero normalization at infinity,

$$L_{ij}\chi \to 0, \qquad \text{at } \lambda \to \infty.$$

Each function $L_{ij}\chi$ satisfies the linear integral equation

$$L_{ij}\chi(\lambda, \bar{\lambda}) = \frac{1}{\pi}\int \frac{L_{ij}\chi(\nu, \bar{\nu}) R(\nu, \bar{\nu}, \mu, \bar{\mu}, x)}{\lambda - \mu} d\nu d\bar{\nu} d\mu d\bar{\mu} \tag{3.20}$$

As far as equation (3.12) is uniqualy resolvable, the homogenous equation (3.21) has only zero solution. Hence,

$$L_{ij}\chi = 0 \tag{3.21}$$

and

$$L_{ij}\chi I_k = 0. \tag{3.22}$$

The linear system (3.17) is just equivalent to system (3.22). Let us substitute asymptotic equation (3.13) in (3.21) and perform an asymptotic expansion of the result. All terms of asymptotic expansion of (3.21) should be identically equal zero. Annigilation of the first nonvanishing term of the order $1/\lambda$ gives system (3.2).

Proof of validity of equations (3.3) is a little more difficult. It will be published in the separate article [3].

4 Integration of the Gauss-Codazzi equations

In R^3 the system (3.2) reads:

$$\frac{\partial Q_{12}}{\partial x_3} = Q_{13}Q_{32} \tag{4.1}$$

$$\frac{\partial Q_{21}}{\partial x_3} = Q_{23}Q_{31} \tag{4.2}$$

$$\frac{\partial Q_{13}}{\partial x_2} = Q_{12}Q_{23} \tag{4.3}$$

$$\frac{\partial Q_{23}}{\partial x_1} = Q_{21}Q_{13} \tag{4.4}$$

$$\frac{\partial Q_{31}}{\partial x_2} = Q_{32}Q_{21} \tag{4.5}$$

$$\frac{\partial Q_{32}}{\partial x_1} = Q_{31}Q_{12} \tag{4.6}$$

To go to the Gauss-Codazzi system one should assume that Q_{ij} do not depend on x_3. Equations (4.1), (4.2) read now

$$Q_{13}Q_{32} = 0, \quad Q_{23}Q_{31} = 0. \tag{4.7}$$

We impose an additional condition compatible with (4.7):

$$Q_{31} = Q_{32} = 0. \tag{4.8}$$

Now equations (4.5), (4.6) are satisfied automatically. Thus, only equations (4.3), (4.4) survived in system (3.2).

System (3.3) for $n = 3$ consistes of three equations:

$$\frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{21}}{\partial x_1} + Q_{13}Q_{23} = 0 \tag{4.9}$$

$$\frac{\partial Q_{13}}{\partial x_3} + \frac{\partial Q_{31}}{\partial x_1} + Q_{12}Q_{32} = 0 \tag{4.10}$$

$$\frac{\partial Q_{23}}{\partial x_3} + \frac{\partial Q_{32}}{\partial x_2} + Q_{21}Q_{31} = 0 \tag{4.11}$$

In virtue of (4.7) equations (4.10), (4.11) are satisfied. Hence the total system of equations, resolving this special type of Gauss-Lamé system is reduced to following equations

$$\frac{\partial Q_{12}}{\partial x_2} = Q_{23} Q_{31}, \quad \frac{\partial Q_{23}}{\partial x_1} = Q_{21} Q_{23}$$
$$\frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{21}}{\partial x_1} + Q_{13} Q_{23} = 0 \tag{4.12}$$

This system should be accomplished by equations for Lamé coefficients,

$$\frac{\partial H_1}{\partial x_2} = Q_{12} H_2, \quad \frac{\partial H_2}{\partial x_1} = Q_{21} H_1. \tag{4.13}$$

System (4.12)-(4.13) coincides with Gauss-Lamé equations (2.11), (2.13). To solve this system one should choose the dressing function $T_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda})$ in a very special way. According to (3.9),

$$R_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda}) = e^{\nu x_i - \lambda x_j}\, \Gamma_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda}). \tag{4.14}$$

To eliminate the dependence of coordinate x_3 in Q_{ij}, one should eliminate this dependence in $R_{ij}(\nu, \bar{\nu}, \lambda, \bar{\lambda})$. Hence, one has to put

$$T_{3i} \simeq \delta(\nu)\delta(\bar{\nu}), \quad T_{i3} \simeq \delta(lambda)\delta(\bar{\lambda}). \tag{4.15}$$

Taking into account the condition (3.11) one can construct T_{ij} by the unique way:

$$\begin{aligned}
T_{12} &= \mu F(\mu, \bar{\mu}, \lambda, \bar{\lambda}), \\
T_{21} &= -\mu F(-\lambda, -\bar{\lambda}, -\mu, -\bar{\mu}), \\
T_{13} &= -\mu f_1(-\mu, -\bar{\mu})\, \delta(\lambda)\delta(\bar{\lambda}), \\
T_{23} &= -\mu f_2(-\mu, -\bar{\mu})\, \delta(\lambda)\delta(\bar{\lambda}), \\
T_{31} &= \mu \delta(\mu)\delta(\bar{\mu})\, f_1(\lambda, \bar{\lambda}), \\
T_{32} &= \mu \delta(\mu)\delta(\bar{=}mu)\, f_2(\lambda, \bar{=}lambda)
\end{aligned} \tag{4.16}$$

To satisfy condition (3.10) one has to put

$$\begin{aligned}
f_1(\lambda, \bar{\lambda}) &= \bar{f}_1(\bar{\lambda}, \lambda), \\
f_2(\lambda, \bar{\lambda}) &= \bar{f}_2(\bar{\lambda}, \lambda), \\
R(\mu, \bar{\mu}, \lambda, \bar{\lambda}) &= \bar{R}(\bar{\mu}, \mu, \bar{\lambda}, \lambda)
\end{aligned} \tag{4.17}$$

Without loss of generality one can put diagonal elements equal zero,

$$T_{11} = T_{22} = T_{33} = 0$$

Formulae for T_{31}, T_{32} include the product $\mu\delta(\mu)\delta(\bar{\mu})$. Normally one has to put this expression zero. Actually, this is true if it is integrated after multiplication to a continous probe function,

$$\int f(\mu, \bar{\mu})\, \mu\delta(\mu)\delta(\bar{\mu}) d\mu d\bar{\mu} = 0, \tag{4.18}$$

but sometimes we will multiply to a function having a simple pole at $\mu = 0$, thus $f(\mu) = g(\mu, \bar{\mu})/\mu$ at $\mu \sim 0$, where g is a continous function. In this case

$$int f(\mu, \bar{\mu})\, \mu\delta(\mu)\delta(\bar{\mu}) d\mu d\bar{\mu} = \int g(\mu, \bar{\mu})\, \delta(\mu)\delta(\bar{\mu}) d\mu d\bar{\mu} = g(0, 0). \tag{4.19}$$

Integral equation (3.12) consists of a family of independent equations imposed separetely on each row of the matrix χ_{ij} (rows are numerated by the first index). Let us consider a system of equations for the third row: $\chi_{31}, \chi_{32}, \chi_{33}$. In virtue of (4.14), (4.16) χ_{33} drops out of the equations for χ_{31}, χ_{32}. As a result, these elements of χ_{ij} satisfy to a homogenous linear system which has only zero solutions. Thus,

$$\chi_{31} \equiv 0, \;\; \chi_{32} \equiv 0, \tag{4.20}$$

and $\chi_{33} \equiv 1$.

Hence, in accordance with (4.8),

$$Q_{31} = Q_{32} = 0, \;\; Q_{33} = 0. \tag{4.21}$$

To find expressions for χ_{13}, χ_{23} one has to use formula (4.17). We have:

$$\begin{aligned} \chi_{13} &= \frac{A}{\lambda} = \frac{Q_{13}}{\lambda}, \;\; \chi_{23} = \frac{B}{\lambda} = \frac{Q_{23}}{\lambda}, && (4.22) \\ A &= Q_{13} = -\frac{1}{\pi} \int \nu \Big[\chi_{11}(\nu, \bar{\nu}, x)\, f_1(-\nu, -\bar{\nu}) e^{\nu x_1} + \chi_{12}(\nu, \bar{\nu}, x) e^{\nu x_2} f_2(-\nu, -\bar{\nu})\Big] d\nu d\bar{\nu}, \\ B &= Q_{23} = -\frac{1}{\pi} \int \nu \Big[\chi_{21}(\nu, \bar{\nu}, x)\, f_1(-\nu, -\bar{\nu}) e^{\nu x_1} + \chi_{22}(\nu, \bar{\nu}, x) e^{\nu x_2} f_2(-\nu, -\bar{\nu})\Big] d\nu d\bar{\nu}. && (4.23) \end{aligned}$$

By the use of (4.20), (4.21) one can reduce system (3.12) to the system of equations imposed on 2×2 matrix,

$$X_{ij} = \chi_{ij}, \quad i = i, 2 \;\; j = 1, 2. \tag{4.24}$$

This system reads:

$$X_{ij}(\lambda,\bar{\lambda},x)=\delta_{ij}+\frac{1}{\pi}\int\frac{X_{ik}(\nu,\bar{\nu},x)e^{\nu x_k-\mu x_j}\,S_{kj}(\nu,\bar{\nu},\mu,\bar{\mu})}{\lambda-\mu}d\nu d\bar{\nu}d\mu d\bar{\mu},\quad (4.25)$$

where a matrix S is a sum of two components,

$$\begin{aligned} &S=U+V,\\ &U=\left\|\begin{array}{cc} 0 & \mu F(\mu,\bar{\mu},\lambda,\bar{\lambda}) \\ -\mu F(-\lambda,-\bar{\lambda},-\mu,-\bar{\mu}) & 0 \end{array}\right\| \quad (4.26)\\ &V_{ij}(\mu,\bar{\mu},\lambda,\bar{\lambda})=-\frac{\mu}{\pi}\,f_i(-\mu,-\bar{\mu})f_j(\lambda,\bar{\lambda}) \quad (4.27)\end{aligned}$$

One should notice that $U(\mu,\bar{\mu},\lambda,\bar{\lambda})$ satisfies the standard relation (3.11), while

$$V^{tr}(-\lambda,-\bar{\lambda},-\mu,-\bar{\mu})=-\frac{\mu}{\lambda}\,V(\mu,\bar{\mu},\lambda,\bar{\lambda}).\quad (4.28)$$

If X_{ij} are known, one can find the Lamé coefficients (the elements of the first quadratic form) by the rules:

$$\begin{aligned} p(x_1,x_2) &= H_1(x_1,x_2)=\int\Big[\chi_{11}(\nu,\bar{\nu},x)e^{\nu x_1}g_1(\nu,\bar{\nu})+\chi_{12}(\nu,\bar{\nu},x)e^{\nu x_2}g_2(\nu,\bar{\nu})\Big]d\nu d\bar{\nu},\\ q(x_1,x_2) &= H_2(x_1,x_2)=\int\Big[\chi_{21}(\nu,\bar{\nu},x)e^{\nu x_1}g_1(\nu,\bar{\nu})+\chi_{22}(\nu,\bar{\nu},x)e^{\nu x_2}g_2(\nu,\bar{\nu})\Big]d\nu d\bar{\nu}.\quad (4.29)\end{aligned}$$

Different choice of g_1, g_2 defines different representatives of the same class of Combescure equivalent surfaces. In particulary, one can put

$$\begin{aligned} g_1(\nu,\bar{\nu}) &= -\frac{\nu}{\pi}f_1(-\nu,-\bar{\nu})\\ g_2(\nu,\bar{\nu}) &= -\frac{\nu}{\pi}f_2(-\nu,-\bar{\nu}).\quad (4.30)\end{aligned}$$

In this case $p=A, q=B$.

Notice that

$$k_1=\frac{A}{p},\;\; k_2=\frac{B}{q},$$

are main curvatures of the surface in a given point. In the case (4.27)

$$k_1=k_2=1,$$

and the surface is a sphere.

We found an interesting fact - each class of Combescure equivalent surfaces includes a sphere. Different classes just define different coordinations of the sphere.

5 Surfaces ot type zero

We will call a function $F(\mu,\bar{\mu},\lambda,\bar{\lambda})$ as a master function of a surface. The surface belongs to a finite type N if its master function can be presented in a form

$$F(\mu,\bar{\mu},\lambda,\bar{\lambda}) sun_{k=1}^{N} a_k(\mu,\bar{\mu}) b_k(\lambda,\bar{\lambda}). \tag{5.1}$$

In this case a solution of the Gauss-Codazzi equation can be found in a closed form. Surfaces of a finite type can be also called "solitonic surfaces". In this article we describe surfaces of zero type, when $F \equiv 0$. In this case

$$S_{ij} = V_{ij} = -\frac{\mu}{\pi} f_i(-\mu,-\bar{\mu})\, f_i(\lambda,\bar{\lambda}) \tag{5.2}$$

One can put

$$\chi_{ij} = \delta_{ij} + \lambda_i(x_1,x_2)\, h_j(\lambda,\bar{\lambda},x_j), \tag{5.3}$$

where

$$h_j(\lambda,\bar{\lambda},x_j) = \frac{1}{\pi}\int \frac{f_i(-\mu,-\bar{\mu})e^{\mu x_j}}{\lambda+\mu} d\mu d\bar{\mu} \tag{5.4}$$

Let us introduce functions

$$c_i(x_i) = \frac{1}{\sqrt{2\pi}}\int f_i(-\mu,-\bar{\mu})e^{\mu x_i}\, d\mu d\bar{\mu}.$$

One can check that

$$\lambda_i = -\frac{\sqrt{2\pi} c_i'(x_i')}{\Delta}, \tag{5.5}$$

$$\Delta = 1 + c_1^2(x_1) + c_2^2(x_2) \tag{5.6}$$

From (5.3) one easily finds:

$$\alpha = Q_{12} = -\frac{2c_1'(x_1)c_2(x_2)}{1+c_1^2(x_1)+c_2^2(x_2)} \tag{5.7}$$

$$\beta = Q_{21} = -\frac{2c_1(x_1)c_2'(x_2)}{1+c_1^2(x_1)+c_2^2(x_2)} \tag{5.8}$$

By substitution (5.3) to (4.21) one obtains

$$A = Q_{13} = -\frac{2c_1'(x_1)}{1+c_1^2(x_1)+c_2^2(x_2)}, \tag{5.9}$$

$$B = -\frac{2c_2'(x_2)}{1+c_1^2(x_1)+c_2^2(x_2)} \tag{5.10}$$

One can check that formulae (5.7)-(5.10) present a solution of the Gauss-Codazzi system.

For elements of matrix ϕ_{ij} one have from (3.16):

$$\begin{aligned}
\phi_{11} &= (1+\lambda_1 h_1)e^{-\lambda x_1}, \\
\phi_{21} &= \lambda_2 h_1 e^{-\lambda x_1}, \\
\phi_{12} &= \lambda_1 h_2 e^{-\lambda x_2}, \\
\phi_{22} &= (1+\lambda_2 h_2)e^{-\lambda x_2}.
\end{aligned} \tag{5.11}$$

To accomplish solution of the Gauss-Codazzi system and find all possible arrays of p, q compatible with (5.7)-(5.10), one should introduce two arbitrary functions

$$\xi_i(\lambda,\bar{\lambda}) = \bar{\xi}_i(\bar{\lambda},\lambda), \quad i = 1, 2.$$

Then p, q will be given by formulae

$$\begin{aligned}
p &= <\xi_1 e^{-\lambda x_1}> + \lambda_1 \left[< h_1 e^{-\lambda x_1} \xi_1 > + < h_1 e^{-\lambda x_1} \xi_2 > \right], \\
q &= <\xi_2 e^{-\lambda x_2}> + \lambda_2 \left[< h_2 e^{-\lambda x_2} \xi_1 > + < h_2 e^{-\lambda x_2} \xi_2 > \right]
\end{aligned} \tag{5.12}$$

In (5.12) the bracets mean just integration by $\lambda\bar{\lambda}$.