Description of the n-Orthogonal Curvilinear Coordinate Systems and Hamiltonian Integrable Systems of Hydrodynamic Type. Part 1. Integration of the Lamé Equations.

Vladimir E. Zakharov[1]
[1] Landau Institute for Theoretical Physics
2 Kosygina Street, Moscow, Russia 117940, zakharov@itp.ac.ru
Department of Mathematics, University of Arizona
Tucson, Az 85721, zakharov@math.arizona.edu

Abstract

The classical problem of description of n-Orthogonal Curvilinear Coordinate System in flat Euclidean space is solved by the Inverse Scattering Method. The developed method allows to describe Hamiltonian and semi-Hamiltonian Integrable systems of Hydrodynamic type.

1 Introduction

The problem of description of n-orthogonal curvilinear coordinate systems can be formulated as follow:

Find in R^n all the coordinate systems:

$$u^i = u^i(x^1, ..., x^n) \tag{1.1}$$

$$\det \| \frac{\partial u^i}{\partial x^j} \| \neq 0 \tag{1.2}$$

satisfying the condition of orthogonality:

$$\sum_{k=1}^{n} \frac{\partial u^i}{\partial x^k} \frac{\partial u^j}{\partial x^k} = 0 \quad i \neq j \tag{1.3}$$

The problem can be formulated either locally (in the same domain Ω) or globally (in the whole R^n). In the later case one can admit that the condition (1.2) can be violated on some manifold of dimension $m < n$, and the system of intersecting hypersurfaces may have a nontrivial topology. Coordinates $u^i(x)$ are defined up to an obvious transformation:

$$u^i = f^i(\tilde{u}^i) \tag{1.4}$$

For $n = 2$ the problem can be solved very easily. Let us choose of a function (u^1, for instance) in an arbitrary way and consider a system of its level lines on the plane x^1, x^2. Then one can construct the vector field of normals to the level lines. Integral curves of this vector field are the level lines for u^2, which can be reconstructed uniquelly up to transformation (1.4).

For $n \geq 3$ the problem is much more difficult. The first nontrivial case $n = 3$ is known in Differential Geometry as the problem of triply orthogonal systems of surfaces. It was formulated in 1810, when Dupin and Binet found a family of confocal quadrics satisfying condition (1.3). Since that time the problem became one of the classical and the most popular. The first general theorem stating that the intersections of two orthogonal surfaces are the lines of curvature was obtained by Dupin in 1813. During more than a century after, the problem was attacked by many first-class mathematicians. Gauss, Lamé, Bonnet, Cayley, Bianchi and Darboux are just the most famous among them. The total amount of published materials, connected to this topic, is enormous. Only the articles of Luigi Bianchi, devoted to the problem of triply-orthogonal systems of surfaces, collected together, comprise the book of 850 pages - the volume 3 of his "Opere", published in Rome in 1955 [1]. The milestone in history of this problem was a fundamental monography "Lecons sur les systems orthogonaux et les cordonees curvilineare" by G. Darboux [2], printed in Paris in 1910. It is really astonishing, how exciting are many pages of this book to a person familiar with the modern mathematical theory of solitons!

After the First World War the problem of n-orthogonal coordinate systems became less popular and temporarily lost its conspicious status. Nevetheless, it attracted the attention of E. Cartan [3], and others, most of them French mathematicians (see, for instance [4,5]).

Let us summarize some basic achievements of the "classical" period. First of all, the problem of n-orthogonal systems of surfaces can be formulated as a problem of intrinsic geometry (Gauss, Lamé). Due to (1,2) one can resolve:

$$x^i = x^i(u^1, ..., u^n) \tag{1.5}$$

Let us denote:

$$H_i^2 = \sum_k \left(\frac{\partial x^i}{\partial u^k} \right)^2 \tag{1.6}$$

The metric tensor in R^n in the coordinate system u^i is diagonal:

$$ds^2 = \sum_{i=1}^{n} H_i^2 (du^i)^2 \tag{1.7}$$

Christoffel's coefficients for the Levi-Civita connection are:

$$\Gamma^i_{lm} = 0 \qquad\qquad i \neq l \neq m$$

$$\Gamma^i_{il} = \frac{1}{H_i} \frac{\partial H_i}{\partial u^l} \tag{1.8}$$

$$\Gamma^i_{ll} = -\frac{H_l}{H_i^2} \frac{\partial H_l}{\partial u^i} \qquad i \neq l \tag{1.9}$$

The space R^n is flat, hence the Riemann's curvature tensor vanishes:

$$R_{il,jm} \equiv 0 \tag{1.10}$$

Becouse the metric tensor is diagonal, the condition (1.10) is satisfied automatically, if:

$$i \neq l \neq j \neq m$$

The conditions:

$$R_{il,im} = 0 \qquad l \neq m \tag{1.11}$$

impose on coefficients H_i (Lamé coefficients) the following system of equations:

$$\frac{\partial^2 H_i}{\partial u^l \partial u^m} = \frac{1}{H_l} \frac{\partial H_l}{\partial u^m} \frac{\partial H_i}{\partial u^l} + \frac{\partial H_m}{H_m \partial H_l} \frac{\partial H_i}{\partial u^m} \tag{1.12}$$

The number of equations (1.12) is $n(n-1)(n-2)/2$.

The conditions:

$$R_{il,il} = 0 \tag{1.13}$$

impose on H_i another system of $n(n-1)/2$ equations:

$$\frac{\partial}{\partial u^l} \frac{\partial H_i}{H_l \partial u^l} + \frac{\partial}{\partial u^i} \frac{\partial H_l}{H_i \partial u^i} + \sum_{m \neq i \neq j} \frac{1}{(H^m)^2} \frac{\partial H_i}{\partial u^m} \frac{\partial H_l}{\partial u^m} = 0 \tag{1.14}$$

The system (1.12), (1.14) is heavily overdeterminated, but still has common solutions. Bianchi [1] and Cartan [3] showed that a general solution of both systems can be parametrized locally by $n(n-1)/2$ arbitrary functions of two variables.

If Lamé coefficients H_i are known, one can find $x^i(u^1, \ldots, u^n)$ (i.e., solve the embedding problem), by solving another overdeterminated (but linear!) problem:

$$\frac{\partial^2 x^i}{\partial u^k \partial u^l} = \Gamma^k_{kl} \frac{\partial x^i}{\partial u^k} + \Gamma^l_{lk} \frac{\partial x^i}{\partial u^l} \tag{1.15}$$

$$\frac{\partial^2 x^i}{\partial (u^l)^2} = \sum_k \Gamma^k_{ll} \frac{\partial x^i}{\partial u^k} \tag{1.16}$$

One can prove (see, for instance the book of Forsythe [6]), that the system (1.15), (1.16) is compatible in virtue of (1.12), (1.14) and defines n-orthogonal surfaces up to transition and orthogonal rotation in R^n (only the case $n = 3$ is considered in this book, but the generalization is easy). It is important to mention, that the system (1.15) alone has much more solutions, which can be parametrized by arbitrary functions of one variable.

An interest to the problem of n-orthogonal surfaces was reestablished a decade ago, when it was found that the problem has natural applications in mathematical physics.

In 1983 Novikov and Dubrovin [7] developed a "geometrical" theory of quasilinear systems of Hydrodynamic type in 1+1 dimensions. These systems have a form:

$$\frac{\partial u^i}{\partial t} = \sum V^i_k(u)\,\frac{\partial u^k}{\partial x} \qquad u = u^1 \cdots u^n \tag{1.17}$$

Novikov and Dubrovin showed that the system (1.17) is a Hamiltonian system with a "local" Hamiltonian:

$$H[u] = \int h(u)dx$$

if the matrix V^i_k can be presented in a form:

$$V^i_k = \sum_k \left(g^{il}(u)\,\frac{\partial^2 h}{\partial u^l \partial u^k} + b^{il}_k(u)\,\frac{\partial h}{\partial u^l} \right) \tag{1.18}$$

Here $g^{il}(u)$ is some metrics in a *flat* space R^n, while:

$$b^{il}_k(u) = -\sum_s g^{ik}\ \Gamma^l_{sk} \tag{1.19}$$

Here Γ^l_{sk} are the corresponding Christoffel's coefficients.

The system (1.17) is a generalization of the Euler equations for ideal compressible fluid (in this case $n = 2$). It was known, since the time of Riemann, that for $n = 2$, the Hamiltonian system (1.17) is integrable by the hodograph method and can be transformed to a diagonal form:

$$\frac{\partial u^i}{\partial t} = V^i(u)\frac{\partial u^i}{\partial x} \tag{1.20}$$

“Diagonal” variables u^i are called Riemann’s invariants, and the coefficients $V^i(u)$ are “diagonal” velocities. In 1984, S. Tzarev, a student of S. Novikov, generalized classical Riemann’s results to the case of arbitrary n[8]. Developing the ideas of Dubrovin and Novikov, he proved that the Hamiltonian system (1.17) can be integrated by some generalization of the hodograph method, only in the case that it can be transformed by a proper choice of variables u^i to the diagonal form (1.20). In this case the flat metrics g^{ik} is diagonal, and the Hamiltonian h satisfies the system of equations:

$$\frac{\partial^2 h}{\partial x^i \partial u^j} = \Gamma^i_{ij} \frac{\partial h}{\partial u^i} + \Gamma^j_{ji} \frac{\partial h}{\partial u^j} \tag{1.21}$$

coinciding to the first half of the embedding conditions (1.15).

Moreover, each solution $P(u)$ of this system generates an integral P of the system (1.17):

$$P = \int P(u) du$$

and all these integrals commute. So, classification of flat diagonal matrixes $ds^2 = H_i^2 du^{i^2}$ is an important preliminary step to classification of integrable Hamiltonian system of hydrodynamic type. To accomplish the classification one must find all solutions of the system coinciding with one of the embedding equations (1.15, 1.16). It is important to mention that the diagonal velocities $V^i(u)$ obey the following overdeterminated system (see [9]):

$$\frac{\partial}{\partial u^i} \left(\frac{1}{V^j - V^k} \frac{\partial V^k}{\partial u^j} \right) = \frac{\partial}{\partial u^j} \left(\frac{1}{V^i - V^k} \frac{\partial V^k}{\partial u^i} \right) \tag{1.22}$$

So, the problems of description of n-orthogonal surfaces and classification of Hamiltonian of hydrodynamic-type systems are almost equivalent. The core of both problems is to find all solutions of overdeterminated system (1.12), (1.14). It is important that the order of these systems can be reduced to one. Let us introduce the ”rotation coefficients” (see, for instance [1,2]):

$$\beta_{ik} = \frac{1}{H_i} \frac{\partial H_k}{\partial u^i} \tag{1.23}$$

¿From (1.12) one can find that β_{ik} satisfy the following first-order system of equations:

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik} \beta_{kj} \tag{1.24}$$

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{m\neq i,j} \beta_{mi}\beta_{mj} = 0 \tag{1.25}$$

If the solution of the system (1.21), (1.22) is known, one can find the Lamé coefficients by solving the linear problem:

$$\frac{\partial \Psi_i}{\partial u^k} = \beta_{ki}\Psi_k \quad i \neq k \tag{1.26}$$

and putting $H_i = \Psi_i$. But a common solution of the system (1.26) is far from to be unique. Let $\tilde{\Psi}_i$ be an another solution. Introducing:

$$V_i = \frac{\tilde{\Psi}_i}{H_i}$$

we get the following identities:

$$\left(\frac{1}{V_k - V_i}\right) \frac{\partial V_i}{\partial u^k} = \frac{\partial \ln H_i}{\partial u^k} \tag{1.27}$$

So, the quantities V_i satisfy the equations (1.22) and are diagonal velocities for some integrable Hamiltonian system of the hydrodynamic type.

Different solutions of the system (1.26), affiliated to a given rotation coefficients β_{ik} describe different n-orthogonal coordinate systems, related by so called Combescure transformation. Suppose H_i and $\tilde{H}_i$ are two sets of Lamé coefficients, related by the Combescure transformation. Their quotient $W_i = H_i/\tilde{H}_i$ satisfy the equations (1.22). System of the hydrodynamic type:

$$\frac{\partial u^i}{\partial \tau} = W_i(u)\frac{\partial u^i}{\partial x} \tag{1.28}$$

is a symmetry of system (1.20). Any set $W_i(u)$ provides a solution $u = u(x,t)$ of (1.20) in an implicit form:

$$\tau W^i(u) = V^i(u)t + x \tag{1.29}$$

A purpose of this article is to show that the systems (1.24) and (1.25) can be integrated by the Inverse Scattering Method (ISM). We will use a version of ISM known as the Dressing Method, formulated by A.B.Shabat and by the author of this paper in 1974 [10] (see also [11]). A starting point of the

Dressing method is construction of a certain integral equation of Marchenko type. Its solution gives exact solutions of (1.24), (1.25) together with fundamental solution of the linear system (1.26). So, it makes possible to find a set of β_{ik} parametrized by $n(n-1)/2$ functions of two variables and construct for a given β_{ik} all n-orthogonal systems related by a Combescure transformation. Each solution of (1.24), (1.25) describes a Hamiltonian system of the hydrodynamic type together with all its symmetries.

Integrability of the system (1.24), (1.25) is not an astonishing fact. In the simplest case $n = 3$ the system (1.24) is nothing but a well-known "three-wave system" (see [10]) on algebra of real 3×3 matrixes l_3. The similiar system on the symmetric space of complex-valued hermitian matrixes is widely used in nonlinear optics. In the general case system (1.24) is a generalization of the three-wave system. It was found in the articles [11,12].

Thus, construction of solution of the system (1.24) is relatively easy problem. The really new and difficult problem is to separate those special solutions of system (1.24) which satisfy the system (1.25) as well. This problem is solved by imposing on the "dressing matrix function" of a certain reduction, which is a differential relation, connecting the dressing matrix with its transponent. We hope that invention of this new type of reductions will allow to find new classes of integrable equations in future.

Integrability of system (1.24), (1.25), though in very restricted sense, was known to the classics in a form of so called "Ribaucour transformation" (see [1]). If a given solution of the systems (1.24), (1.25), (1.26) is known, one can find a new solution by the formula:

$$\hat{\beta}_{ik} = \beta_{ik} - \frac{2\Psi_i}{A}\left(\frac{\partial \Psi_k}{\partial u^k} + \sum_{s \neq k} \beta_{sk} \Psi_k\right) \tag{1.30}$$

Here $A = \sum_p (\Psi_p)^2$.

We will show that the Ribaucour transformation is a very special case of the dressing procedure, which allows to use for dressing a complete set of $n(n-1)/2$ arbitrary functions of two variables.

Mathematicians of the classical period found a number of special solutions of systems (1.12), (1.14), (1.24), (1.25). One of the most remarkable is so called Egorov's solutions:

$$H_i^2 = \frac{\partial \Phi}{\partial u^i} \tag{1.31}$$

Here Φ is some scalar function of u^i. In this case the matrix of rotational coefficients is symmetric:

$$\beta_{ik} = \beta_{ki}$$

and the system (1.22) can be reduced to the form:

$$\sum_{k=1}^{n} \frac{\partial}{\partial u^k} \beta_{ij} = 0 \qquad i \neq j \tag{1.32}$$

Also system (1.21), (1.22) can be reduced to a n-wave system on the algebra of real matrixes in $1+1$ dimensional space. This fact has been established by B. Dubrovin [13], who also applied Egorov's metrics to classification of Frobenius manifolds in topological quantum field theory [15].

In addition we must remark that the problem of n-orthogonal surfaces can be formulated in a loosened form. Namely, we can impose on the Lamé coefficients only equations (1.12) and drop out equations (1.14). The obtaned system describes n-orthogonal metrics in a Riemann space of a special type, defined by the condition of "diagonality" of the Riemann's curvature tensor:

$$R_{iklm}(1 - \delta_{il}\delta_{km}) = 0 \tag{1.33}$$

We will call such Riemann spaces *spaces of diagonal curvature*. In Hydrodynamics it corresponds to a system of hydrodynamic type, which can be diagonal (and hence is integrable), but has no local Hamiltonian structure, *semi-hamiltonian system*. We will shoa that construction of such systems is an easier problem than construction of integrable Hamiltonian systems.

In this article we consider construction of n-orthogonal systems as a problem of intrinsic geometry and basically don't touch ahe problem of embeddina. According to this attitude, we don't try to find Hamiltonians and conservation laws to find systems of the Hydrodynamic type. We are going to discuss this problem in the next article. Also we construct n-orthogonal systems locally in some domains in R^n, and don't discuss so far the problem of globalization. All obtained results can be easily expanded to the case of pseudo-Euclidean space $R^{(p,q)}$.

2 The Dressing Method and the Abstract n-Wave System

In this chapter we describe a method of solution of the system (1.12) and equivalent systems (1.24), (1.25). The system (1.24) is a special case of a much more general integrable system, which can be written in a rather abstract form.

Let A be an associitiative algebra over a field of real numbers R or complex numbers C. Let Ω be a domain in R^n and $u = (u^1, \dots, u^n)$ are coordinates in R^n. Then $\Omega(u)$ is a A-valued function on Ω. We introduce n A-valued functions on Ω, $I_k(u)$, which commute:

$$[I_i, I_k] = 0 \tag{2.1}$$

and obey the condition:

$$\frac{\partial I_k}{\partial u^i} = 0 \qquad i \neq k \tag{2.2}$$

So I_k depends only on u^k. In typical cases I_k are constants.

We consider now the system of $n(n-1)(n-2)/6$ overdeterminated systems of nonlinear equations imposed on Q:

$$\sum_{perm} \epsilon_{ijk} \, (I_i \frac{\partial Q}{\partial u^j} I_k - I_i Q I_j Q I_k) = 0 \tag{2.3}$$

Here ϵ_{ijk} is antisymmetric with respect to all permutation, and:

$$\epsilon_{ijk} = 1 \qquad i > j > k \tag{2.4}$$

Summation in (2.3) is going over all possible six permutations.

We call the system (2.3) the *abstract n-wave system.* It was introduced in [11,12] for matrix algebras as a natural generalization of well-known n-wave system.

If I_i are constant in A, they are defined up to an arbitrary linear transformation:

$$I_k = \sum_k q_{ik} \, \tilde{I}_k \qquad det\|q_{ik}\| \neq 0$$

The system (2.3) is invariant with respect to the transformation

$$Q \rightarrow Q + J \qquad [I_k, J] = 0$$

Let $A_0 < A$ be the maximum commutative subalgebra containing all I_k. One can decompose A in a sum of linear spaces

$$A = A_0 + A_1$$

Without loss of generality one can consider that $Q \in A_1$.

Suppose that E is the unit element in A and $I_n = E$. (If the algebra A has no unit element, one can interprete E as a unit operator $E : A \rightarrow A$)

Then the system of $(n-1)(n-2)/2$ equations in (2.3), associated with I_n, can be written in the form:

$$\frac{\partial}{\partial u^j}[I_i, Q] - \frac{\partial}{\partial u^i}[I_j, Q] + I_i \frac{\partial Q}{\partial u^n} I_j - I_j \frac{\partial Q}{\partial u^n} I_i - [[I_i, Q], [I_j, Q]] = 0 \qquad (2.5)$$

$$(i \neq j \neq n)$$

This is a "standard form" of the n-wave system (see, for instance, [11]).

It is important to mention, that solving of the system (2.5), it is enough to solve all other equations, included to (2.3). To be sure in this, one can multiply (2.5) to I_k, $k \neq i \neq j \neq n$, from the left side and do the cyclic permutation. As a result, one achieves all remained equations in (2.3). We proved the following simple theorem:

Theorem 2.1

Equations (2.3), $i \neq j \neq k \neq n$ are compatibility conditions for equations (2.5).

In the general case none of I_k is the unit in A. But we can use the proved theorem to construct solutions of (2.3) at arbitrary I_k. We just make Q dependent on an auxilliary variable $u^{n+1} = s$, $-\infty < s < \infty$, and put $I_{n+1} = E$.

Then we consider the system:

$$\frac{\partial}{\partial u^j}\,[I_i, Q] - \frac{\partial}{\partial u^i}\,[I_j, Q] + I_i\,\frac{\partial Q}{\partial s}\,I_j - I_j\,\frac{\partial Q}{\partial s}\,I_i - [[I_i, Q], [I_j, Q]] = 0 \quad (2.6)$$

We know that:
Any solution of the system (2.8) at any fixed s is automatically a solution of the system (2.3).

The system (2.6) can be solved by the dressing method. The main tool here is the integral equation:

$$K(s, s', u) = F(s, s', u) + \int_s^\infty K(s, q, u)F(q, s', u)dq \tag{2.7}$$

Here $K, F \in A$ and $-\infty < s < \infty$, $-\infty < s' < \infty$.

Suppose that $F(s, s', u)$ is a given function satisfying two conditions:

1. Equation (2.7) is uniquelly resolved.
2. $F(s, s', u)$ obeys the set of equations:

$$D_i F = \frac{\partial F}{\partial u^i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i = 0 \tag{2.8}$$

Then $Q = K(s, s, u)$ obeys system (2.8) and consequantly the system (2.3).
The proof of this fact is straightforward. One can write equation (2.7) in a symbolic form:

$$K = F + K * F \tag{2.9}$$

and denote:

$$D_i F = \frac{\partial F}{\partial u^i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i \tag{2.10}$$

Applying D_i to (2.10) one get after simple transformations:

$$\tilde{D}_i K = D_i F + \tilde{D}_i K * F + K * D_i F \tag{2.11}$$

$$\tilde{D}_i K = \frac{\partial K}{\partial u^i} + I_i \frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'} I_i + [I, Q]K \tag{2.12}$$

As far as $D_i F = 0$, we have:

$$\tilde{D}_i K = \tilde{D}_i(K) * F \tag{2.13}$$

Then, in virtue of unique resolvability of equation (2.7) one receives:

$$\tilde{D}_i K = 0 \tag{2.14}$$

Here :

$$\tilde{D}_i K = \frac{\partial K}{\partial u^i} + I_i \, \frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'} \, I_i + [I_i, Q] K \tag{2.15}$$
$$Q = K(s, s, u)$$

Then we set:

$$[\tilde{D}_i, \tilde{D}_j] K = 0 \tag{2.16}$$

An operator $[\tilde{D}_i, \tilde{D}_j]$ is a multiplication from the left to some element $R_{ij}(s, u) \in A$. The equation (2.16) holds identically for all s', hence one can cancel K. We get:

$$[\tilde{D}_i, \tilde{D}_j] = 0 \tag{2.17}$$

So, the operators $\tilde{D}_i$ commute. One can check that condition (2.17) coincides identically with (2.6).

Let Ψ_0 be any solution of the system:

$$\frac{\partial \Psi_0}{\partial u^i} + I_i \, \frac{\partial \Psi_0}{\partial s} = 0 \tag{2.18}$$

Then $\Psi(s, u)$ is defined as follow:

$$\Psi = \Psi_0(s, u) + \int_s^\infty K(s, s', u) \, \Psi_0(s', u) ds' \tag{2.19}$$

A direct calculation shows that Ψ satisfies the system:

$$L_i \Psi = 0$$

$$L_i \Psi = \frac{\partial \Psi}{\partial u^i} + I_i \, \frac{\partial \Psi}{\partial s} + [I_i, Q] \Psi \tag{2.20}$$

Moreover, L_i commute:

$$[L_i, L_j] = 0 \tag{2.21}$$

It is just another notation of the identity (2.17).

The system of linear equations (2.20) is compatible. Its compatibility conditions are equations (2.6). Thus, (2.20) gives a “Lax representation” for the system (2.6). One can construct the Lax representation for system (2.3) as well. It is given by linear equations

$$(I_j L_i - I_i L_j)\Psi = 0 \tag{2.22}$$

or:

$$I_j \frac{\partial \Psi}{\partial u^i} - I_i \frac{\partial \Psi}{\partial u^j} = (I_i Q I_j - I_j Q I_i)\Psi \tag{2.23}$$

The procedure for construction of exact solution of the system (2.3) is a generalization of the ”Dressing Method”, introduced in [10]. This is the simpliest case of dressing - the dressing against a “trivial background”. We can essentially generalize this procedure, considering the dressing against an arbitrary solution of equations (2.6).

Let $Q_0(u,s)$ be such a solution. Suppose that K and F are connected as previousely by the relation (2.9) and F satisfies the system of equations:

$$D_i[Q_0]F = \frac{\partial F}{\partial u^i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'} I_i + [I_i, Q_0(s,u)]F - F[I_i, Q_0(s',u)] = 0 \tag{2.24}$$

In virtue of (2.3) this system is compatible. Applying operators $D_i[Q_0]$ to the equation (2.9) we get:

$$\begin{aligned} \tilde{D}_i[Q]K &= 0 \\ \tilde{D}_i[Q]K &= \frac{\partial K}{\partial u^i} + I_i \frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'} I_i + \\ &+ [I_i, Q_0(s,u)]K - K[I_i, Q_0(s',u)] = 0 \qquad (2.25) \\ \tilde{Q} &= Q_0 + K(s,s',u) \qquad (2.26) \end{aligned}$$

The equation (2.25) can be solved if one can find a fundamental solution of the linear system:

$$\frac{\partial \Psi_0}{\partial u^i} + I_i \frac{\partial \Psi_0}{\partial s} + [I_i, Q_0(s,u)]\ \Psi_0 = 0 \tag{2.27}$$

Then:

$$F = \Psi_0(s,u)\, F_0(s,s',u)\, \Psi_0^{-1}(s',u) \tag{2.28}$$

Here F_0 satisfies the system (2.8). To find Ψ one must apply the transformation (2.19) to any solution of (2.27).

Apparently:

$$\delta Q(s,u) = F(s,s,u) \tag{2.29}$$

at any s satisfy the linearized system (2.3):

$$\sum \epsilon_{ijk} \, (I_i \, \frac{\partial}{\partial u^j} \delta Q \, I_k - I_i \, Q_0 \, I_j \, \delta \, Q \, I_k - I_i \delta \, Q \, I_j \, Q_0 \, I_k) = 0 \tag{2.30}$$

and the linearized system (2.6):

$$\begin{aligned} &\frac{\partial}{\partial u^i}[I_i, \delta Q] - \frac{\partial}{\partial u^i}[I_i, \delta Q] + I_i \frac{\partial}{\partial s} \delta Q I_j - I_j \frac{\partial}{\partial s} \delta Q I_i + \\ &+ [[I_i, \delta Q], [I_j, Q_0]] + [[I_i, Q_0], [I_j, \delta Q]] = 0 \end{aligned} \tag{2.31}$$

Finding of Ψ if $Q_0(s,u)$ is known, is a solution of "the direct scattering problem" (in terms of the theory of solitons).

The dressing method gives explicit solutions of the system (2.3), if the kernel F is degenerative:

$$F = \sum_{q=1}^{N} f_q(s,u) \;\, g_q(s',u) \tag{2.32}$$

Here f_q, g_q satisfy the equations:

$$L_i \;\, f_q = 0 \qquad\quad L^+ \;\, g_q = 0 \tag{2.33}$$

and L_i^+ is an adjoint to L_i operator:

$$\text{Ł}_i^+ g = \frac{\partial g}{\partial u^i} + \frac{\partial g}{\partial s'} \; I_i - g \, [I_i, Q_0] = 0 \tag{2.34}$$

If $N = 1$ and:

$$F(s,s',u) = f(s,u) \;\, g(s',u) \tag{2.35}$$

then:

$$
\begin{aligned}
K(s,s',u) &= K(s,u)\ g(s',u) \\
K(s,u) &= f(s,u)\left[1-\int_s^\infty g(s',u)\ f(s',u)\,ds'\right]^{-1} && (2.36) \\
K(s,s,u) &= f(s,u)\left[1-\int_s^\infty g(s',u)\ f(s',u)ds'\right]^{-1}\ g(s,u) && (2.37)
\end{aligned}
$$

Presumed unique solvability of the equation (2.9) is a garantee that the inversion in (2.36), (2.37) is possible.

3 n - Orthogonal Systems in Spaces of Diagonal Curvature and Semi-Hamiltonian Systems of Hydrodynamic Type

Let us show, how the construction of Section 2 works for systems (1.12), (1.23), (1.24). In this case A is $l_n(R)$, that means the algebra of $n\times n$ matrixes with real coefficients. I_k are diagonal matrixes. They can be choosen as follow:

$$I_k = diag(\overbrace{\underbrace{0,\ldots,1}_{k}\,0,\ldots,0}^{n}) \tag{3.1}$$

Obviousely:

$$I_i\, I_k = 0 \tag{3.2}$$

if $i \neq k$.

It is immediately clear that system (2.3) has now a form:

$$\frac{\partial Q_{ik}}{\partial u^j} = Q_{ij}\ Q_{jk} \tag{3.3}$$

that formally coincides with (1.24). Here $i \neq j \neq k$. In future we will identify $Q_{ik} = \beta_{ki}$. Then:

$$\left(\frac{\partial}{\partial u^i} + \frac{\partial}{\partial u^j} + \frac{\partial}{\partial s}\right) Q_{ij} - \sum_{k \neq ij} Q_{ik} Q_{kj} = 0 \quad i \neq j \tag{3.4}$$

The linear equation (2.22) is simplified up to the form:

$$I_i \, \frac{\partial \Psi}{\partial u^i} = I_i \, Q \, I_j \, \Psi \tag{3.5}$$

Let Ψ_i is any column in the matrix Ψ. The (3.2) gives:

$$\frac{\partial \Psi_i}{\partial u^j} = Q_{ij} \, \Psi_j \qquad i \neq j \tag{3.6}$$

For $i = j$ one gets from (2.20):

$$\frac{\partial \Psi_i}{\partial u^i} + \frac{\partial \Psi_i}{\partial s} + \sum_{k \neq j} Q_{ik} \; \Psi_k = 0 \tag{3.7}$$

The equation (2.8) can be solved as follow:

$$F_{ij}(s, s', u) = f_{ij}(s - u^i, s' - u^j) \tag{3.8}$$

Here $f_{ij}(\xi, \eta)$ are n^2 arbitrary functions of two variables.

Substituting (3.3) to (3.4), and (3.6) to (3.7) yields:

$$\frac{\partial Q_{ij}}{\partial s} + D \, Q_{ij} = 0 \qquad \frac{\partial \Psi_i}{\partial s} + D \, \Psi_i = 0 \tag{3.9}$$

Here:

$$D = \sum_{k=1}^{n} \frac{\partial}{\partial u^k} \tag{3.10}$$

So:

$$\begin{aligned} \Psi_i &= \Psi(u^1 - s, \ldots, u^n - s) \\ Q_{ij} &= Q_{ij}(u^1 - s, \ldots, u^n - s) \end{aligned} \tag{3.11}$$

In this case dependence of the auxilliary parameter s is very simple.

Now we can describe a procedure of implementation the Dressing Method to the formulated problem. It consists of the following steps:

1. Choose arbitrarily a real matrix function of two variable $f_{ij}(\xi, \eta)$ and solving the integral equation (2.9). The only restriction on f_{ij} - the equation (2.9) must be uniquelly resolvable for $-\infty < s < \infty$, $a \in \Omega$.

   Any choice of f_{ij} produces *one-parameter family* of solutions of the system (3.3) by formula:

   $$Q_{ij}(s, u) = K_{ij}(s, s, u) \tag{3.12}$$

   The solutions are parametrized by $-\infty < s < \infty$. Dependence upon s according to (3.11) is just a shift of arguments $u^i \to u^i - s$.

2. Choosing arbitrarily a solution of system (2.18), one constructs one solution of the system (3.6) using the "Dressing Formula" (2.19). Each column of Ψ_0 in (2.19) is dressed independently, so one can parametrize Ψ_i by an arbitrary vector solutions of (2.18). In our case they have a very simple form:

   $$\Phi_{0i} = \phi_i(s - u^i) \tag{3.13}$$

   Here $\phi_i(\xi)$ is an arbitrary vector function of one variable.

3. Now we can identify:

   $$H_i = \Psi_i \tag{3.14}$$

We receive a solution of the system (1.12) describing an intrinsic geometry of some space of diagonal curvature. Another choice of $\phi_i(\xi)$ produce another solution of (1.12), $\tilde{H}_i$, related to (3.14) by a Combescure transformation. All possible quotients:

$$V_i(u) = \frac{\tilde{H}_i}{H_i} \tag{3.15}$$

define an integrable semi-hamiltonian system (1.20). The other choice of H_i and $\tilde{H}_i$, that is H_i' and $\tilde{H}_i'$, generates another semi-hamiltonian system:

$$\tilde{W}_i(u) = \frac{\tilde{H}_i'}{H_i'} \tag{3.16}$$

The systems:

$$\begin{aligned}\frac{\partial u^i}{\partial t} &= V_i(u)\,\frac{\partial u^i}{\partial x}\\ \frac{\partial u^i}{\partial \tau} &= W_i(u)\,\frac{\partial u^i}{\partial x}\end{aligned}\tag{3.17}$$

are compatible and they are symmetries of each other. Each $W_i(u)$ generates in implicit form an exact solution of the systems (1.20) (see [9]):

$$\tau V_i(u) = tW_i(u) + x \tag{3.18}$$

To accomplish this subject let us write the complete set of equations imposed on K_{ij}:

$$\frac{\partial K_{ij}}{\partial u^k} = Q_{ik}K_{kj} \quad k \neq i \neq j \tag{3.19}$$

$$\frac{\partial K_{ij}}{\partial u^i} + \frac{\partial K_{ij}}{\partial s} + \sum_{k\neq i} Q_{ik}K_{kj} = 0 \quad i \neq j \tag{3.20}$$

$$\frac{\partial K_{ij}}{\partial u^j} + \frac{\partial K_{ij}}{\partial s'} - Q_{ij}K_{ji} = 0 \quad i \neq j \tag{3.21}$$

$$\frac{\partial K_{ii}}{\partial u^i} + \frac{\partial K_{ii}}{\partial s} + \frac{\partial K_{ii}}{\partial s'} + \sum_{k\neq i} Q_{ik}K_{ki} = 0 \tag{3.22}$$

$$\frac{\partial K_{ii}}{\partial u^j} = Q_{ij}K_{ji} \quad i \neq j$$

¿From (3.19), (3.20) one derives:

$$\frac{\partial K_{ij}}{\partial s} + DK_{ij} = 0 \quad i \neq j \tag{3.23}$$

$$\frac{\partial K_{ii}}{\partial s} + \frac{\partial K_{ii}}{\partial s'} + DK_{ii} = 0$$

Hence :

$$\begin{aligned} K_{ij}(s,s',u) &= K_{ij}(s',u^1-s,\ldots,u^n-s) \quad i\neq j \\ K_{ii}(s,s',u) &= K_{ii}(s'-s,u^1-s,\ldots,u^n-s) \end{aligned} \tag{3.24}$$

Now we can study the dressing against an arbitrary background. Suppose, one solution $H_{0i}(u)$ of (1.12) is known. Then one can find $Q_{0ij}(u)$ by:

$$Q_{ij} = \frac{1}{H_{0j}}\,\frac{\partial H_{0i}}{\partial u^j} \tag{3.25}$$

and extend it for all values of s by relation (3.11):

$$Q_{0ij}(u^1,\ldots,u^n) \to Q_{0ij}(u^1-s,\ldots,u^n-s) = Q_0(u-s) \tag{3.26}$$

The equation (1.25) takes now the form:

$$\frac{\partial F}{\partial u^i} + I_i\frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'}\,I_i + [I_i,Q_0(u-s)]F - F[I_i,Q_0(u-s')] = 0 \tag{3.27}$$

To solve this equation one must solve the equation (2.27). It is enough to find *a fundamental* solution Ψ_0 of the system:

$$\frac{\partial \Psi_{0i}}{\partial u^j} = Q^{(u)}_{0ij}\;\Psi_{0j} \quad i\neq j \tag{3.28}$$

and to extend it to all s by (3.11):

$$\Psi_0(u^1,\ldots,u^n) \to \Psi_0(u^1-s,\ldots,u^n-s) = \Psi_0(u-s) \tag{3.29}$$

Then one must use (2.28), where:

$$F_{0ij}(s,s',u) = f_{0ij}(s-u^i,s'-u^j) \tag{3.30}$$

and $f_{0ij}(\xi,\eta)$ are arbitrary. One column in the matrix Ψ_0 is given by H_{0i}. All others are connected to H_{0i} by a Combescure transformation. So, finding of all Combescure-equivalent metrics to a given one is actually the solution of the direct scattering problem.

4 Algebraic Reductions

The machinery built up in previous chapters, makes it possible to construct exact solutions of the system (2.3). It is unclear so far how dense the set of the found solution is and how efficiently one can approximate by this solution a generic solution of (2.3). This question is especially difficult if the conditions of periodicity or quasi-periodicity are imposed. We will not discuss that really important question in this article. In a sense we constructed too many solutions, and we concentrate our efforts in retrieving some interesting special classes of them.

Let * mean an involution in A:

$$(a^*)^* = a \qquad a, b \in A \qquad (ab)^* = b^* a^* \tag{4.1}$$

If A is l_n, this involution might be:

$$a^* = R^{-1} a^{tr} R \tag{4.2}$$

Here a^{tr} is the matrix transponent to a. R is an arbitrary matrix satisfying the condition $R^{tr} = \pm R$.

The following theorem holds:

Theorem 4.1

Let F and K be connected by relation:

$$K = F + K \times F \tag{4.3}$$

and $\epsilon(s)$ is an A- valued function on s satisfying the condition $\epsilon^*(s) = \pm\epsilon(s)$.

Let:

$$F(s, s')\epsilon(s') = \epsilon(s) F^*(s', s) \tag{4.4}$$

Then:

$$K(s, s)\epsilon(s) = \epsilon(s) K^*(s, s) \tag{4.5}$$

<u>Proof</u>

Let us expand K in powers of F and present:

$$K(s,s) = F(s,s) + \int_s^\infty F(s,q)F(q,s)dq + \cdots \tag{4.6}$$

Then:

$$K^*(s,s) = F^*(s,s) + \int_s^\infty F^*(q,s)F^*(s,q)dq + \cdots \tag{4.7}$$

¿From (4.4) one can see that:

$$F(s,s)\epsilon(s) = \epsilon(s)F^*(s,s)$$

Let us multiply (4.6) from the right and (4.7) from the left to $\epsilon(s)$ and compare the results. Applying (4.4) to any term in the expansion (4.7) one can realize that they coincide.

Identities (4.4), (4.5) present an *Algebraic reduction* imposed on Marchenko equation (4.3). In the simplest case it takes the form:

$$F(s,s') = F^*(s',s) \tag{4.8}$$

Then:

$$K(s,s) = K^*(s,s) \tag{4.9}$$

Let us study algebraic reduction in the abstract n-wave system. Starting from the simpliest reduction (4.8), (4.9) we assume that the involution * remains all I_k unchanged.

$$I_k^* = I_k \tag{4.10}$$

Then the constrain (4.4) is compatible with the equation (2.8). Imposing of this constrain provides that in (2.3):

$$Q^* = Q \tag{4.11}$$

This reduction defines special classes of solutions in (2.3).

To make a degenerative kernel F satisfying the reduction (4.8) one must put:

$$F \;=\; F_1 + F_2 \tag{4.12}$$

$$F_1 = \sum_{q=1}^{N_1} f_q(s,u) f_q^*(s',u) \tag{4.13}$$

$$F_2 = \sum_{p=1}^{N_2} \Big(f_p(s,u) g_p^*(s',u) + g_p(s,u) f_p^*(s',u) \Big) \tag{4.14}$$

If $A = l_n$ is defined by (3.1), one can take involution in the form (4.2), where R is a diagonal matrix:

$$R = diag\epsilon_i \qquad \epsilon_i = \epsilon_i(u^i) \tag{4.15}$$

In particular one can put:

$$R = E \qquad Q^* = Q^{tr}$$

Now:

$$Q_{ik} = Q_{ki} \qquad \frac{1}{H_i}\,\frac{\partial H_k}{\partial u^i} = \frac{1}{H_k}\,\frac{\partial H_i}{\partial u^k} \tag{4.16}$$

and:

$$H_i^2 = \frac{\partial \Lambda}{\partial u^i} \tag{4.17}$$

Here $\Lambda = \Lambda(u)$ is a scalar function obeying the following overdeterminated system of third-order equations:

$$\begin{aligned} &\frac{\partial \Lambda}{\partial u^i}\,\frac{\partial \Lambda}{\partial u^l}\,\frac{\partial \Lambda}{\partial u^k}\,\frac{\partial^3 \Lambda}{\partial u^i \partial u^l \partial u^k} = \\ &= \frac{1}{4}\left(\frac{\partial \Lambda}{\partial u^i}\,\frac{\partial^2 \Lambda}{\partial u^l \partial u^i}\,\frac{\partial^2 \Lambda}{\partial u^k \partial u^i} + \frac{\partial \Lambda}{\partial u^l}\,\frac{\partial^2 \Lambda}{\partial u^i \partial u^l}\,\frac{\partial^2 \Lambda}{\partial u^k \partial u^k} + \right. \\ &\left. + \frac{\partial \Lambda}{\partial u^k}\,\frac{\partial^2 \Lambda}{\partial u^l \partial u^k}\,\frac{\partial^2 \Lambda}{\partial u^k \partial u^i} \right) \end{aligned} \tag{4.18}$$

It is clear that the system (4.18) is compatible and integrable. Its integrating can be done by the dressing method, described in Sections (2,3). To provide fulfilling of reduction (4.15) one must impose that the matrix f_{ij} in (3.7) satisfies the symmetry conditions:

$$f_{ij}(\xi,\eta) = f_{ji}(\eta,\xi) \tag{4.19}$$

Let us study what kind of general algebraic reductions (4.4), (4.5) are possible in the system (2.3), (2.6). Suppose that condition (4.10) is satisfied. The reduction must be compatible with the basic equation (2.6). It is done if:

$$[I_k, \epsilon] = 0 \tag{4.20}$$

and:

$$(\frac{\partial}{\partial u^i} + I_i \frac{\partial}{\partial s})\epsilon = 0 \tag{4.21}$$

For n-orthogonal system (3.3), (3.4) it implies that ϵ is a diagonal matrix:

$$\epsilon_{ij} = diag \epsilon_i$$

$$\epsilon_i = \epsilon_i(u^i - s)$$

and:

$$f_{ij}(\xi, \eta) = \frac{\epsilon_i(\eta)}{\epsilon_j(\xi)} f_{ji}(\eta, \xi) \tag{4.22}$$

Now:

$$Q_{ik}(u) = \frac{\epsilon_i(u^j)}{\epsilon_k(u^i)} Q_{ki}(u) \tag{4.23}$$

It is easy to check that reduction (4.19) is compatible with the system (2.3)

The described algebraic reductions are common in the theory of solitons (see, for instance [11], [16]).

5 Differential Reductions

In this chapter we introduce a new class of reductions in integrable systems, the differential reductions. These reductions are very essential for integration of n-orthogonal curvilinear coordinate systems, and for the theory of integrable systems of hydrodynamic type. We will start with the following:

Theorem 5.1

Let A be an associative algebra, * be an involution in A, $F(s,s')$ be an A-valued function of two variables $-\infty < s < \infty$, $-\infty < s' < \infty$, and the equation:

$$K(s,s') = F(s,s') + \int_s^\infty K(s,q)F(q,s')dq \tag{5.1}$$

be resolvable uniquely for all s.

Let F satisfy the equation:

$$\frac{\partial F(s,s')}{\partial s'} + \frac{\partial F^*(s',s)}{\partial s} = 0 \tag{5.2}$$

Then on the diagonal $s' = s$:

$$\left(\frac{\partial K}{\partial s'} + \frac{\partial K^*}{\partial s}\right)|_{s=s'} = K(s,s)\,K^*(s,s) \tag{5.3}$$

Proof:

Expand $K(s,s')$ in series in powers of F:

$$K(s,s') = F(s,s') + \cdots + \int_s^\infty \cdots \int_s^\infty F(s,q_1)F(q_1,q_2)\cdots F(q_n,s')dq_1\cdots dq_n + \cdots \tag{5.4}$$

Then:

$$K^*(s',s) = F^*(s',s) + \cdots + \int_{s'}^\infty \cdots \int_{s'}^\infty F^*(q_1,s)F^*(q_1,q_2)\cdots F^*(q_n,s')dq_1\cdots dq_n + \cdots \tag{5.5}$$

(We renamed in (5.4) $q_1,\ldots,q_n \to q_n,\ldots,q_1$)

Let us denote:

$$R(s) = \lim_{s'\to s}\left(\frac{\partial K(s,s')}{\partial s'} + \frac{\partial K^*(s',s)}{\partial s}\right) \tag{5.6}$$

Expanding R in powers of F, F^* one gets:

$$R = R_2 + \cdots + R_n + \cdots \tag{5.7}$$

$$
\begin{aligned}
R_n &= \int_s^\infty \cdots \int_s^\infty F(s,q_1)\cdots F(q_{n-2},q_{n-1})\frac{\partial F(q_{n-1},s')}{\partial s'}dq_1\cdots dq_{n-1}\\
&+ \int_{s^*}^\infty \cdots \int_{s^*}^\infty \frac{\partial F^*(q_1,s)}{\partial s}F^*(q_1,q_2)\cdots F^*(q_{n-1},s)dq_1\cdots dq_{n-1} \quad (5.8)
\end{aligned}
$$

The first term in (5.7) drops out due to (5.2).

Now one can apply to the n-st term in (5.7) the identity (5.2) $n-1$ times and perform $n-1$ integrations by parts. One can see that the terms containing $n-1$ integrations, cancel. Terms, including $n-2$ integrations, can be collected in the sum:

$$
\begin{aligned}
R_n &= F(s,s)\int_s^\infty \cdots \int_s^\infty F^*(q_1,s)\cdots F^*(q_{n-1},s)dq_2\cdots dq_{n-1} + \cdots +\\
&+ \int_s^\infty \cdots \int_s^\infty F(s,q_1)\cdots F(q_{n-2},s)dq_1\cdots dq_{n-2}F^*(s,s) \quad (5.9)
\end{aligned}
$$

Then all the sum in (5.7) presents the expansion for the product $K(s,s)K^*(s,s)$. The theorem is proven.

The constraint in (5.2) is the simpliest example of a differential reduction. If F is a generative kernel, one can present it (see (4.10), (4.12)) like:

$$
\begin{aligned}
F &= F_1 + F_2 && (5.10)\\
F_1 &= \sum_{q=1}^{N_1} \frac{\partial p_q^*(s,u)}{\partial s}\Lambda f_q(s',u) && (5.11)\\
F_2 &= \sum_{p=1}^{N_2} \left(\frac{\partial h_p^*(s,u)}{\partial s}g_p(s',u) - \frac{\partial g_p^*(s,u)}{\partial s}h_p(s',u)\right) && (5.12)\\
\Lambda^* &= -\Lambda
\end{aligned}
$$

Here f_p, g_p, h_p are arbitrary A-valued functions.

The theorem 5.1 can be generalized further. Let L be an A-valued differential operator:

$$
LF = \sum_{n=0}^{N} \frac{\partial^n F(s,s')}{\partial s'^n}U_n(s') \quad (5.13)
$$

While $\tilde{L}$ is the ajoint operator:

$$\tilde{L}F = \sum_{n=0}^{N}(-1)^n \frac{\partial^n}{\partial s^n} U_n(s) F \tag{5.14}$$

Suppose that $F(s, s')$ satisfies the equation:

$$LF(s, s') = \tilde{L}F^*(s', s) \tag{5.15}$$

Applying to (5.15) the involution * and permutating $s \leftrightarrow s'$, one derives:

$$\tilde{L}^* F(s, s') = L^* F^*(s', s) \tag{5.16}$$

Here:

$$L^* F^* = \sum_{n=0}^{N} U_n^* \frac{\partial^n F^*(s', s)}{\partial s^n} \tag{5.17}$$

$$\tilde{L}^* F = \sum_{n=0}^{\infty} (-1)^N \frac{\partial^n}{\partial s'^n} F(s, s') U_n^*(s') \tag{5.18}$$

Comparing (5.15) and (5.16), one derivess:

$$\tilde{L}^* = \pm L \tag{5.19}$$

The condition (5.19) imposes $n+1$ relations on coefficients of L:

$$\begin{aligned} U_n^*(s) &= \pm U_n(s) \\ U_{n-1}^*(s) &= \mp \left(U_{n-1} + U'_n(s)\right) \end{aligned} \tag{5.20}$$

A relation (5.15) can be called a differential reduction of the order n. Algebraic reduction is a trivial case of differential reductions (the order of the operator L is zero).

Let us consider:

$$\left(LK(s, s') - \tilde{L}K^*(s', s)\right)|_{s=s'} \tag{5.21}$$

One can show that R can be expressed through:

$$K(s, s')|_{s=s'} \ , \ K^*(s', s)|_{s'=s}$$

and a finite number of derivatives:

$$\frac{\partial^q K(s,s')}{\partial s^q}|_{s=s'}\ ,\ \frac{\partial^q K(s,s')}{\partial s'^q_n}|_{s=s'}\ ,\ \frac{\partial^q K^*(s',s)}{\partial s'^q}|_{s=s'}\ ,\ \frac{\partial^q K^*(s',s)}{\partial s^q}|_{s=s'}\ (q<n)$$

Moreover, R is a bilinear operator (linear with respect to $K(s,s'), K^*(s',s)$ separetely).

We are not going to prove here this fact in its general form. We just will display two simplest examples.

1. Let:

$$LF = \frac{\partial F(s,s')}{\partial s'}U(s') + F(s,s')V(s')$$

and:

$$\begin{aligned} U^*(s) &= U(s) \\ V(s)+V^*(s) &= U'(s) \end{aligned} \tag{5.22}$$

Now:

$$R = K(s,s)U(s)K^*(s,s) \tag{5.23}$$

and the relation (5.21) takes the following form:

$$\begin{aligned} \left[\frac{\partial K(s,s')}{\partial s'}U(s') + \frac{\partial}{\partial s}U(s)K^*(s',s)\right]_{s=s'} &+ K(s,s)V(s) - V(s)K^*(s,s) = \\ &= K(s,s)U(s)K^*(s,s) \end{aligned} \tag{5.24}$$

In particular, if:

$$U = 1 \quad V^*(s) = -V(s) \tag{5.25}$$

one derives:

$$\begin{aligned} \lim_{s'\to s}\left(\frac{\partial K(s,s')}{\partial s'} + \frac{\partial}{\partial s}K^*m(s',s)\right) &+ K(s,s)V(s) - V(s)K^*(s,s) = \\ &= K(s,s)K^*(s,s) \end{aligned} \tag{5.26}$$

2. Let us choose:

$$LF = \frac{\partial^2 F(s,s')}{\partial s'^2}$$

Using the same technique of expansion in series one can show that in this case the relation (5.21) becomes:

$$\begin{aligned}&\left(\frac{\partial^2 K(s,s')}{\partial s'^2} - \frac{\partial^2 K^*(s',s)}{\partial s^2}\right)|_{s=s'} = \\ &\left(\frac{\partial K(s,s')}{\partial s} K(s,s) - K(s,s)\frac{\partial K^*(s,s')}{\partial s}\right)|_{s=s'} \qquad (5.27)\end{aligned}$$

Now we will show how the simplest differential reduction (5.2), (5.3) works for the abstract n-wave system. Suppose, that $I_k^* = I_k$ and consider the identity:

$$I_j D_i K - I_i D_j K = 0 \qquad (5.28)$$

which is:

$$\begin{aligned}I_j \frac{\partial K(s,s')}{\partial u^i} - I_j \frac{\partial K(s,s')}{\partial u^j} \ &+\ I_j \frac{\partial K(s,s')}{\partial s'} I_i - I_i \frac{\partial K(s,s')}{\partial s'} I_j + \\ &+\ (I_j Q I_i - I_i Q I_j) K(s,s') = 0 \qquad (5.29)\end{aligned}$$

Applying to (5.29) the involution * and permutating $s \leftrightarrow s'$, one gets:

$$\begin{aligned}\frac{\partial K^*(s,s')}{\partial u^j} I_i - \frac{\partial K(s',s)}{\partial u^i} I_j \ &+\ I_j \frac{\partial K^*(s',s)}{\partial s} - I_i \frac{\partial K^*(s',s)}{\partial s} I_j + \\ &+\ K^*(s',s)(I_j Q^* I_i - I_i Q^* I_j) = 0 \qquad (5.30)\end{aligned}$$

Adding (5.27), (5.28), putting $s' \to s$, and using the relation (5.3), one gets after a simple calculation:

$$\begin{aligned}I_j \frac{\partial Q}{\partial u^i} - I_i \frac{\partial Q}{\partial u^j} \ &+\ \frac{\partial Q^*}{\partial u^j} I_i - \frac{\partial Q^*}{\partial u^i} I_j + I_j Q Q^* I_i - I_i Q Q^* I_j - \\ -I_j Q I_i Q \ &+\ I_i Q I_j Q - Q^* I_j Q^* I_i + Q^* I_i Q^* I_j = 0 \qquad (5.31)\end{aligned}$$

6 n-Orthogonal coordinate systems in the flat space-dressing against a cartesian background

Now we can apply the Dressing Method to the Lamé equations (1.12), (1.14). Again, A is $l_n(R)$ and I_k are given by (3.1). According to Section 3, the dressing function $F(s,s',u)$ is given by the expression (3.8). Let us assume, that it satisfies also the simplest differential reduction (5.2), which can be written as follow:

$$\frac{\partial F_{ij}(s,s',u)}{\partial s'}+\frac{\partial F_{ji}(s',s,u)}{\partial s}=0 \tag{6.1}$$

To find additional equations imposed by (6.1) on Q_{ij} one can use the general formula (5.29) or just apply the involution * and the permutation $s \leftrightarrow s'$ to the equation (3.21). One can read:

$$\begin{aligned}\frac{\partial K_{ij}(s,s',u)}{\partial u^j} &+ \frac{\partial K_{ji}(s',s,u)}{\partial u^i}+\frac{\partial K_{ij}(s,s',u)}{\partial s'}+\frac{\partial K_{ji}(s',s)}{\partial s}- \\ &- Q_{ij}(s)K_{jj}(s,s')-K_{ii}(s,s')Q_{ij}(s')=0\end{aligned} \tag{6.2}$$

The relation (5.3) has now the form:

$$\left(\frac{\partial K_{ij}(s,s')}{\partial s'}+\frac{\partial K_{ji}(s',s)}{\partial s}\right)|_{s=s'}=\sum_l K_{il}(s,s)K_{jl}(s,s) \tag{6.3}$$

Putting in (6.2) $s'=s$ and using (6.3) yields immediately:

$$\frac{\partial Q_{ij}}{\partial u^j}+\frac{\partial Q_{ji}}{\partial u^i}+\sum_{l\neq i,j} Q_{il}Q_{jl}=0 \tag{6.4}$$

The equation (6.4) is identical to (1.25) (remember, that $Q_{ij}=\beta_{ji}$!). To resolve the equations (3.8), (6.1) one can introduce $n(n-1)/2$ functions of two variables $\Phi_{ij}(\xi,\eta)$, $i<j$ and put:

$$\begin{aligned}F_{ij} &= \frac{\partial \Phi_{ij}(s-u^i,s'-u^j)}{\partial s} \qquad i<j \\ F_{ji} &= \frac{\partial \Phi_{ji}(s'-u^i,s-u^j)}{\partial s}\end{aligned} \tag{6.5}$$

Equations (3.8), (6.1) are satisfied now for all off-diagonal elements $i \neq j$. To satisfy the diagonal elements one has to introduce n diagonal antisymmetric functions of two variables:

$$\Phi_{ii}(\xi, \eta) = -\Phi_{ii}(\eta, \xi)$$

and put:

$$F_{ii} = \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s} \tag{6.6}$$

We found that our solution is parametrized by $n(n-1)/2$ functions of two variables $\Phi_{ij}(\xi, \eta)$ together with n additional antisymmetric functions $\Phi_{ii}(\xi, \eta)$. So, the total number of functional parameters participating in the dressing procedure, is even more then the needed number $n(n-1)/2$. That means that in reality we constructed certain classes of equivalent dressings. This equivalence will be considered in the another article.

The expansion (5.10) now has the form:

$$\begin{aligned}
F &= F^{(1)} + F^{(2)} \\
F^{(1)}_{ij} &= \sum_{p=1}^{N_1} \frac{\partial f^{(p)}_{ki}(s - u^i)}{\partial s} \Lambda^{(p)}_{kl} f^{(p)}_{lj}(s' - u^j) \qquad (6.7) \\
F^{(2)}_{ij} &= \sum_{p=1}^{N_2} \left(\frac{\partial h^{(p)}_{ki}(s - u^i)}{\partial s} g^{(p)}_{kj}(s' - u^j) - \frac{\partial g^{(p)}_{ki}(s - u^i)}{\partial s} h^{(p)}_{kj}(s' - u^j) \right)
\end{aligned}$$

Here $f^{(p)}, g^{(p)}, h^{(p)}$ are arbitrary real matrix functions of one variable, while:

$$\Lambda^{(p)}_{kl} = -\Lambda^{(p)}_{lk} \tag{6.8}$$

are arbitrary constant antisymmetric matrixes.

The simplest solution, obtained by the dressing method, appears if $F^{(2)} = 0$, $N_1 = 1$, $f^{(1)}$ is diagonal, and $f_{ij} = f^{(1)}_{ij} = f_i(s - h^i)\delta_{ij}$.

Now:

$$F_{ij} = \frac{\partial f_i(s - u^i)}{\partial s} \Lambda_{ij} g_j(s' - u^j) \qquad \Lambda_{ij} = -\Lambda_{ji} \tag{6.9}$$

Assuming:

$$K_{ij} = w_{ij}(s,u) f_j(s' - u^j) \tag{6.10}$$

we observe that integration in (2.7) can be done explicitly. One derives:

$$w_{ij}(s,u) = \frac{\partial f_i}{\partial s}(s - u^i) \sum_k \Lambda_{ik} (T^{-1})_{kj} \tag{6.11}$$

Here T^{-1} is the inverse matrix to:

$$\begin{aligned} T_{ij} &= \delta_{ij} + \frac{1}{2} f_i^2 \Lambda_{ij} && (6.12) \\ Q_{ij} &= w_{ij}(s,u) f_j(s - u^j) && (6.13) \end{aligned}$$

Now we can put:

$$H_i(s,u) = \phi(s - u^i) + \sum_j w_{ij}(s,u) \int_s^\infty f_j(s' - u^j) \phi_j(s' - u^j) ds' \tag{6.14}$$

Here $\phi_i(\xi)$ is n arbitrary function of one variable. The expression (6.14) presents the simpliest explicit solution of Lamé equations, while (6.13) gives exact solutions of the system (1.24), (1.25). Different choice of $\phi_i(\xi)$ provides the Combescure covariance.

In the simplest case $n = 3$:

$$w_{ij} = -\frac{1}{\triangle} \frac{\partial f_i}{\partial u^i} (\Lambda_{ij} + \frac{1}{2} \Lambda_{ik} \Lambda_{jk} f_k^2) \quad k \neq i, j \tag{6.15}$$

Here:

$$\triangle = \det \| T_{ij} \| = 1 + \frac{1}{4} \sum_{i<j} \Lambda_{ij}^2 f_i^2 f_j^2 \tag{6.16}$$

Now one can change $f_i \to N f_i$ and put $N \to \infty$. Q_{ij} remaines finite in this limit. Now:

$$Q_{ij} \to -2 \frac{\partial f_i}{\partial u^i} f_i \Lambda_{ik} \Lambda_{jk} f_k^2 \left(\sum_{i<j} \Lambda_{ij}^2 f_i^2 f_j^2 \right)^{-1} \tag{6.17}$$

Let us denote:

$$\Psi_k = (\Lambda_{ik}\Lambda_{jk}f_k)^{-1} \quad i \neq j \neq k \;\; i < j \tag{6.18}$$

The expression (6.17) can be rewritten as follow:

$$Q_{ij} = -\frac{2}{A}\frac{\partial \Psi_i}{\partial u^i}\Psi_j \quad A = \sum_p \Psi_p^2 \tag{6.19}$$

The expression (6.19) can be interpreted as follow. Suppose an n-orthogonal system is a decartian system of orthogonal planes. Now:

$$Q_{ik} = 0 \qquad H_i = \Psi_i(u^i)$$

and Ψ_i is an arbitrary function of one variable.

Comparing (6.19) to (1.30) one can see that we constructed the Ribaucour transformation against the simpliest cartesian background. The whole procedure described above is dressing on the cartesian background.

Indeed, at $s \to \infty$ we have $Q_{in} \to 0$, and our n-orthogonal coordinate system goes to a system of orthogonal hyperplanes.

7 Dressing on an arbitrary background

Suppose now that the array $H_{0i}(u)$ satisfies both Lamé equations (1.12), (1.14). To realize the dressing procedure, starting from this solution, one have to find Q_{0ij} by (3.25) and to extand it to all s by (3.26). According to Section 3, any solutions of the system (3.27) give a solution of (1.12). To satisfy also the system (1.14) one has to find a generalization of the differential reduction (5.2), (6.1), compatible with (3.27). The answer is given by the following:

Theorem 7.1

The dressing function $F(s, s', u)$, satisfying the condition (3.27):

$$\frac{\partial F}{\partial u^i} + I_i\frac{\partial F}{\partial s} + \frac{\partial F}{\partial s'}I_i + [I_i, Q_0(s)]F - F[I_i, Q_0(s')] = 0 \tag{7.1}$$

gives the solution of the system (1.14) if it satisfies also the differential reduction:

$$\frac{\partial F(s,s',u)}{\partial s'}+\frac{\partial F^{tr}(s',s,u)}{\partial s} + F(s,s',u)\left[Q_0(u,s')-Q^{tr}(u,s')\right] - \\ - \left[Q_0(u,s)-Q_0^{tr}(u,s)\right]F^{tr}(s',s)=0 \quad (7.2)$$

To prove the theorem one must first check the compatability of (3.27) and (7.1). It is enough to apply the operator D_i to (7.1) and put the result zero in virtue of (3.27) and (7.1). It imposes on Q_0 the following equation:

$$\frac{\partial}{\partial u^k}(Q_0-Q_0^{tr})+I_k\frac{\partial Q_0}{\partial s}-\frac{\partial Q_0^{tr}}{\partial s}I_k = \\ = Q_0^{tr}I_kQ_o^{tr}+Q_0I_kQ_0-Q_0Q_0^{tr}I_k+I_kQ_0Q_0^{tr}+(Q_0^{tr})^2I_k-I_kQ_0^2 \quad (7.3)$$

Let put in (7.2) $Q_0 = Q_{0ij}$. The condition (7.2) is antisymmetric and satisfied automatically for $i = j$. Let $i \neq j \neq k$. Now (7.2) is satisfied due to Q_{0ij} satisfies the equation (3.3).

Let $i \neq k = j$. Then:

$$\frac{\partial}{\partial u^k}(Q_{0ik}-Q_{0ki})+\frac{\partial Q_{0ki}}{\partial s}-Q_{0kk}(Q_{0ki}-Q_{0ik})-\sum_l Q_{0kl}Q_{0il}+\sum_l Q_{0kl}Q_{0li} = \\ = -\sum_{l\neq k\neq j} Q_{0kl}Q_{0il}+\sum_{l\neq k\neq j} Q_{0kl}Q_{0li} = -\sum_{l\neq k\neq j} Q_{0kl}Q_{0il}+\sum_{l\neq k\neq j}\frac{\partial Q_{0ki}}{\partial u^l} \quad (7.4)$$

or:

$$\frac{\partial Q_{0ik}}{\partial u^k}+\frac{\partial Q_{0ki}}{\partial s}-\sum_{l\neq k}\frac{\partial Q_{0ki}}{\partial u^l}+\sum_{l\neq k\neq i} Q_{0kl}Q_{0il}=0 \quad (7.5)$$

Now, using the condition (3.9):

$$\frac{\partial Q_{0ki}}{\partial s}=\sum_l\frac{\partial Q_{0ki}}{\partial u^l} \quad (7.6)$$

one gets:

$$\frac{\partial Q_{0ik}}{\partial u^k} + \frac{\partial Q_{0ki}}{\partial u^i} + \sum_{l \neq k \neq i} Q_{0kl} Q_{0il} = 0 \tag{7.7}$$

The equation (7.3) is identical to (6.4).

The rest proof is straghtforward. Due to the results of Section 5, on the diagonal $s = s'$:

$$\begin{aligned} &\left(\frac{\partial K_{ij}(s,s')}{\partial s'} + \frac{\partial K_{ji}(s',s}{\partial s} \right) |_{s=s'} + \\ &+ \sum_l \left(K_{il}(Q_{0lj} - Q_{0jl} - (Q_{0il} - Q_{0li})K_{jl} \right) = \sum_l K_{il} K_{jl} \end{aligned} \tag{7.8}$$

¿From (3.27) one can find that $K(s,s')$ satisfies the equation:

$$\frac{\partial K}{\partial u^j} + I_j \frac{\partial K}{\partial s} + \frac{\partial K}{\partial s'} I_j + [I_j Q(s,u)]K - K[I_j Q_0(u,s)] \tag{7.9}$$

where:

$$Q = Q_0 + K(s,s)$$

¿From (7.9) one derives:

$$\frac{\partial K_{ij}(s,s')}{\partial u^j} + \frac{\partial K_{ij}(s,s')}{\partial s'} - Q_{ij}(s) K_{ij}(s,s') + \sum_{l \neq j} K_{il}(s,s') Q_{0lj}(s') = 0 \tag{7.10}$$

Permutating in (7.10) $i \leftrightarrow j$ and $s \leftrightarrow s'$ we have:

$$\frac{\partial K_{ji}(s,s')}{\partial u^i} + \frac{\partial K_{ji}(s',s)}{\partial s} - Q_{ji}(s') K_{ii}(s',s) + \sum_{l \neq i} K_{jl}(s',s) Q_{0li}(s) = 0 \tag{7.11}$$

Now we can put in (7.11), (7.12) $s = s'$. Combining then (7.9), (7.11), (7.12) and (7.7) one receives after elementary calculations the following:

$$\frac{\partial Q_{ij}}{\partial u^j} + \frac{\partial Q_{ji}}{\partial u^I} + \sum_{l \neq i \neq j} Q_{il} Q_{jl} = 0 \tag{7.12}$$

This accomplishes the proof. One can mention that:

$$F = \Psi_0 F_0 \Psi_0^{-1}$$

is a common solution of the equations (3.27), (7.1), if F_0 realizes dressing against a cartesian background, Ψ_0 is orthogonal:

$$\Psi_0^{-1} = \Psi_0^{tr}$$

and

$$\frac{\partial \Psi_0}{\partial s} = (Q_0 - Q_0^{tr})\Psi_0 \tag{7.13}$$

It is unclear if Ψ_o can be choosen orthogonal for a general case.

8 Egorov's metrics

Let us find what kind of algebraic reductions are possible in the equation describing n-orthogonal systems. According to Section 4, all algebraic reductions in a space of diagonal curvature are given by (4.20). Substituting (4.21) to (6.4) and using (3.3) yields the following:

$$\begin{aligned}
&\frac{\partial}{\partial u^k} Q_{ik} + \epsilon_k(u^k) \frac{\partial}{\partial u^i} \frac{1}{\partial \epsilon_i(u^i)} Q_{ik} + \sum_{l \neq i,n} Q_{il} Q_{kl} = \\
&= \left(\frac{\partial}{\partial u^k} + \epsilon_k \frac{\partial}{\partial u^i} \frac{1}{\partial \epsilon_i} \right) Q_{ik} + \epsilon_k \sum \frac{1}{\epsilon_l} Q_{il} Q_{lk} = \\
&= \left(\frac{\partial}{\partial u^k} + \epsilon_k \frac{\partial}{\partial u^i} \frac{1}{\epsilon_i} + \epsilon_k \sum \frac{1}{\epsilon_l} \frac{\partial}{\partial u^l} \right) Q_{ik} = 0
\end{aligned} \tag{8.1}$$

Replacing $i \leftrightarrow k$ and using (4.21), we find:

$$\left(\frac{\partial}{\partial u^i} + \epsilon_i \frac{\partial}{\partial u^k} \frac{1}{\epsilon_k} + \epsilon_i \sum \frac{1}{\epsilon_l} \frac{1}{\partial Q_l} \right) \frac{\epsilon_k}{\epsilon_i} Q_{ik} = 0 \tag{8.2}$$

Comparing (8.1) and (8.2) we derive:

$$\epsilon_i = \epsilon_k = const$$

Then the only possible algebraic reduction is:

$$Q_{ik} = Q_{ki} \tag{8.3}$$

In this case:

$$\frac{\partial Q_{ik}}{\partial s} = 0 \qquad \sum_l \frac{\partial Q_{ik}}{\partial u^l} = 0 \tag{8.4}$$

The reduction (8.3) defines so called "Egorov's metrics". In this case:
h

$$Hh_i^2 = \frac{\partial \Lambda}{\partial u^i} \tag{8.5}$$

ahnd:

$$\Omega_{ik} = \frac{1}{2} \frac{\partial^2 \Lambda}{\partial u^i \partial u^k} \left(\frac{\partial \Lambda}{\partial u^i} \frac{\partial \Lambda}{\partial u^k} \right)^{-1/2} \tag{8.6}$$

Here $\Lambda(u)$ is a scalar function, satisfying the equation (4.16) together with the system of equations (8.4). The dressing function F in the Egorov's case satisfies the conditions:

$$f_{ij}(\xi, \eta) = f_{ji}(\eta, \xi)$$

$$\frac{\partial f_{ij}(\xi, \eta)}{\partial \eta} + \frac{\partial f_{ji}(\eta, \xi)}{\partial \xi} = 0$$

or:

$$\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \right) f_{ij}(\xi, \eta) = 0 \tag{8.7}$$

Finally we have:

$$f_{ij} = f_{ij}(\xi - \eta)$$

or:

$$f_{ij} = f_{ji} = f_{ij}(s - s' - u^i + u^j) \tag{8.8}$$

We can assume $F_{ii} = 0$. So Egorov's metrics are parametrized by
$n(n-1)/2$ real functions of one variable.

In the Egorov's case the problem of n-orthogonal coordinate systems is reduced to the following system of first-order equations:

$$\begin{aligned} \frac{\partial Q_{ij}}{\partial u^k} &= Q_{ik} Q_{kj} \qquad Q_{ki} = Q_{ik} \\ \sum_l \frac{\partial Q_{ik}}{\partial u^l} &= 0 \end{aligned} \tag{8.9}$$

In the case $n = 3$ it is a system of three equations:

$$\begin{aligned}&\frac{\partial Q_{12}}{\partial u^1}+\frac{\partial Q_{12}}{\partial u^2}=-Q_{13}Q_{23}\\&\frac{\partial Q_{13}}{\partial u^2}=Q_{12}Q_{23}\\&\frac{\partial Q_{23}}{\partial u^1}=Q_{12}Q_{13}\end{aligned}\tag{8.10}$$

This is the classical system of "three-waves" in the case of exact resonance (Q_{ij} are real). It has standard solitonic solutions (see, for instance [11]). In a general case (8.9) is a system of n-waves (actually $n(n-1)/2$!). This fact was established first by B.Dubrovin [13].

In the Egorov's case the kernel F in the integral equation (2.7) depends only on difference $s-s'$. So, it belongs to the Wiener-Hopf class, and its solution is equivalent to the solution of a certain matrix Riemann-Hilbert's problem. Now on the diagonal $Q=\lim_{s\to s'}K(s,s')$ does not depend on s.

One can use Egorov's metrics as a background for dressing. Now Q_0 in (3.27) does not depend on s, and the equation (1.2) is reduced to (6.1).

The result of dressing is Egorov's metrics itself, if the dressing function is symmetric.

9 Pseudo-Euclidean metrics

All the results obtained above can be easily extended to n-orthogonal curvilinear coordinate systems in $(p+q)$-dimensional flat pseudo-Euclidean space $R^{p,q}$, with the metrics:

$$ds^2=\sum_{i=1}^{p+q}\epsilon_i(du^i)^2 \tag{9.1}$$

Here:

$$\begin{aligned}\epsilon_i &= -1 \qquad i=1,\dots,p\\ \epsilon_i &= 1 \qquad i=p+1,\dots,p+q\end{aligned}\tag{9.2}$$

To find the equations describing n-orthogonal metrics one can use the following formal trick. First we can consider H_i, U_i, Q_{ij} complex. Then we return to real numbers, assuming:

$$u^k \to iu^k \quad Q_{jk} \to -iQ_{jk} \qquad k = 1, \ldots, p$$
$$u^k \to u^k \quad Q_{jk} \to Q_{jk} \qquad k = p+1, \ldots, p+q$$

Now:

$$Q_{jk} = \epsilon_k \frac{\partial H_j}{H_k \partial U_k} \tag{9.3}$$

The system (3.3) is invariant with respect to the transformation (9.3). The system (6.4) after the transformation takes the form:

$$\epsilon_j \frac{\partial Q_{ij}}{\partial u^h j} + \epsilon_i \frac{\partial Q_{ji}}{\partial u^i} + \sum_{lh \neq i,j} Q_{il} Q_{jl} \epsilon_l = 0 \tag{9.4}$$

As far as the system (3.3) is huntouched, the dressing is realized by a matrix function F, satisfying the equathion (2.8) and having a form (3.8). All the difference with the Euclidean case is the differential reduction. It must be taken in the form:

$$\frac{\partial F(s, s')}{\partial s'} R + R \frac{\partial F^{tr}(s', S)}{\partial s} = 0 \tag{9.5}$$

Here R is the diagonal matrix $R = diag \epsilon_i$.

Indeed, from (5.24) one gets:

$$\left(\frac{\partial K_{ij}(s, s')}{\partial s'} \epsilon_j + \epsilon_i \frac{\partial K_{ji}(s', s)}{\partial s} \right)_{s=s'} = \sum_l K_{il} K_{jl} \epsilon_l \tag{9.6}$$

Now, expressing derivatives by s' and s in (9.7) from (3.21), one receives the system (9.4).

The reduction (9.4) realizes the dressing on a cartesian background. To perform the dressing on an arbitrary background, one must change the reduction to:

$$\begin{aligned} &\frac{\partial F(s, s')}{\partial s'} R + R \frac{\partial F^{tr}(s', s)}{\partial s} + F(s, s') \Big(R Q_0(u, s') - Q_0^{tr}(u, s') R \Big) - \\ &- \Big(R Q_0(u, s) - Q_0^{tr}(u, s) R \Big) F^{tr}(s', s) = 0 \end{aligned} \tag{9.7}$$

The system (3.3), (9.4), (9.8) allows the following algebraic reduction:

$$RQ = Q^{tr}R$$

or:

$$\epsilon_i Q_{ij} = \epsilon_j Q_{ji} \tag{9.8}$$

¿From (9.4) one derives now:

$$\frac{\partial H_j}{H_k \partial u^k} = \frac{\partial H_k}{H_j \partial u^j}$$

and:

$$H_i^2 = \frac{\partial \Phi}{\partial u^i}$$

This is Egorov's metrics in the pseudo-Euclidean space.

10 Conclusion

We see that the Lamé equation, describing n-orthogonal curvilinear coordinate systems in flat Euclidian and pseudo-Euclidian spaces can be solved efficiently by using the simplest known version of the dressing method. By solving these equations we achieve also a description of integrable Hamiltonian systems of hydrodynamic type together with their symmetries. So far unsolved is the problem of embedding of obtained metrics to the coordinate space (finding $x^i(u)$) and the equivalent problem of consrtucting of hamiltonians and conserving quantities for the corresponding system of hydrodynamic type. We hope to solve this problem by using more sophysticated versions of the dressing method, based on implementation of the $\bar{\delta}$-problem on complex plane [14]. It will be a subject of the second part of this article. Our results can be generalized by several natural ways. Let us outline the most obvious directions of this generalization and the program of future research. The following problems can be solved in the near future:

1. Exploration of highest differential reductions, allowed by the systems of Lamé equation. These reductions are:

$$L(\frac{\partial}{\partial s'})F(s,s') = L^+(\frac{\partial}{\partial s})F^{tr}(s',s) \tag{10.1}$$

Here:

$$L = \sum_{n=0}^{N} a_n \frac{\partial^{2n}}{\partial s^{2n}} \qquad L^+ = L$$

or:

$$L1983m_{n=1}^{N} a_n \frac{\partial^{2n+1}}{\partial s^{21983}} \qquad L^+ = -L$$

are 1983 operators with constant scaliar coefficients. The reductions (10.1) are further generalization of Egorov's reductions. They impose on Q complicate systems of nonlinear differential relations, which deserve to be studied.

2. Transition to other associative algebras A. The algebra of complex-valued matrixes is the closest natural object. Other interesting objects are infinite dimentional algebras of integral and differential operators. Reductions (both algebraic and differential) in such algebras can generate very interesting new classes of integrable systems.

3. All the developed techniques can be extended to the non-commutative version of the Lamé equations [see [14]):

$$\frac{\partial^2 H_i}{\partial u^l \partial u^m} = \frac{\partial H_l}{\partial u^m} H_l^{-1} \frac{\partial H_i}{\partial u^l} + \frac{\partial H_m}{\partial u^l} H_m^{-1} \frac{\partial H_i}{\partial u^m} \qquad (10.2)$$

Here H_i are non-commuting matrixes. Classification of reduction in the system (10.2) is especially interesting.

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